$$\frac{\textbf{(HW grade)} \cdot \mathbf{0.5} + \textbf{(Test grade)} \cdot \mathbf{0.2}}{\mathbf{0.7}}$$

0, 1-Polynomials

$$f_0(x) = 1$$
$$f_1(x) = 1 + x^3$$
$$f_2(x) = 1 + x^3 + x^{15}$$
$$f_3(x) = 1 + x^3 + x^{15} + x^{16}$$
$$f_4(x) = 1 + x^3 + x^{15} + x^{16} + x^{32}$$
$$f_5(x) = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33}$$
$$f_6(x) = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34}$$
$$f_7(x) = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34} + x^{35}$$

Problem: Prove that this sequence is infinite.

Definitions and Notations: Let $f(x) \in \mathbb{C}[x]$ with $f(x) \not\equiv 0$. Define $\tilde{f}(x) = x^{\deg f} f(1/x)$. The polynomial $\tilde{f}$ is called the *reciprocal* of $f(x)$. The constant term of $\tilde{f}$ is always non-zero. If the constant term of f is non-zero, then $\deg \tilde{f} = \deg f$ and the reciprocal of $\tilde{f}$ is f. If $\alpha \neq 0$ is a root of f, then $1/\alpha$ is a root of $\tilde{f}$. If $f(x) = g(x)h(x)$ with $g(x)$ and $h(x)$ in $\mathbb{C}[x]$, then $\tilde{f} = \tilde{g}\tilde{h}$. If $f = \pm\tilde{f}$, then f is called *reciprocal*. If f is not reciprocal, we say that f is *non-reciprocal*. If f is reciprocal and α is a root of f, then $1/\alpha$ is a root of f. The product of reciprocal polynomials is reciprocal so that a non-reciprocal polynomial must have a non-reciprocal irreducible factor. For $f(x) \in \mathbb{Z}[x]$, we refer to the *non-reciprocal part of* $f(x)$ as the polynomial $f(x)$ removed of its irreducible reciprocal factors having a positive leading coefficient. For example, the non-reciprocal part of $3(-x+1)x(x^2+2)$ is $-x(x^2+2)$ (the irreducible reciprocal factors 3 and $x-1$ have been removed from the polynomial $3(-x+1)x(x^2+2)$).

Lemma 9.1.1. *Let $f(x)$ be an arbitrary polynomial in $\mathbb{Z}[x]$. If the non-reciprocal part of $f(x)$ is reducible, then there exist polynomials $u(x)$ and $v(x)$ in $\mathbb{Z}[x]$ satisfying $u(x)$ and $v(x)$ are both non-reciprocal and $f(x) = u(x)v(x)$.*

Lemma 9.1.2. *Let $f(x) \in \mathbb{Z}[x]$ with $f(0) \neq 0$, and suppose $f(x) = u(x)v(x)$ where each of $u(x)$ and $v(x)$ is non-reciprocal. Then the polynomial $w(x) = u(x)\tilde{v}(x)$ has the following properties:*

(i) $w(x) \neq \pm f(x)$ *and* $w(x) \neq \pm \tilde{f}(x)$.

(ii) $w(x)\widetilde{w}(x) = f(x)\tilde{f}(x)$.

(iii) $w(1)^2 = f(1)^2$.

(iv) $\|w\| = \|f\|$.

Lemma 9.1.3. *Suppose $f(x)$ is a $0,1$-polynomial with $f(0) \neq 0$ and $f(x) = u(x)v(x)$ where each of $u(x)$ and $v(x)$ is non-reciprocal and each of $u(x)$ and $v(x)$ has a positive leading coefficient. Then the polynomial $w(x) = u(x)\tilde{v}(x)$ also has the following properties:*

(i) $w(x) \neq \pm f(x)$ *and* $w(x) \neq \pm \tilde{f}(x)$.

(ii) $w(x)\widetilde{w}(x) = f(x)\tilde{f}(x)$.

(iii) $w(1)^2 = f(1)^2$.

(iv) $\|w\| = \|f\|$.

(v) $w(x)$ *is a* $0,1$*-polynomial with the same number of non-zero terms as* $f(x)$.

(vi) $w(1) = f(1)$.

$$F(x) = u(x)v(x), \quad w(x) = u(x)\tilde{v}(x)$$

$u(x)$ and $v(x)$ are non-reciprocal

(v) if F is a $0,1$-polynomial, then w is also and with the same number of non-zero terms as F

$$F(x) = \sum_{j=1}^{r} a_j x^{d_j}, \quad w(x) = \sum_{j=1}^{s} b_j x^{e_j}$$

$$\left(\sum_{j=1}^{s} b_j\right)^2 \le \left(\sum_{j=1}^{s} b_j^2\right)^2 = \left(\sum_{j=1}^{s} a_j^2\right)^2$$

$$= \left(\sum_{j=1}^{s} a_j\right)^2 = \left(\sum_{j=1}^{s} b_j\right)^2$$

Lemma 9.1.3. *Suppose $f(x)$ is a $0,1$-polynomial with $f(0) \neq 0$ and $f(x) = u(x)v(x)$ where each of $u(x)$ and $v(x)$ is non-reciprocal and each of $u(x)$ and $v(x)$ has a positive leading coefficient. Then the polynomial $w(x) = u(x)\tilde{v}(x)$ also has the following properties:*

(i) $w(x) \neq \pm f(x)$ *and* $w(x) \neq \pm \tilde{f}(x)$.

(ii) $w(x)\widetilde{w}(x) = f(x)\tilde{f}(x)$.

(iii) $w(1)^2 = f(1)^2$.

(iv) $\|w\| = \|f\|$.

(v) $w(x)$ *is a $0,1$-polynomial with the same number of non-zero terms as* $f(x)$.

(vi) $w(1) = f(1)$.

Examples of questions we would like to answer:

1. How does

$$f(x) = 1 + x^{211} + x^{517} + x^{575} + x^{1245} + x^{1398}$$

factor in $\mathbb{Z}[x]$?

2. Let $f_0(x) = 1$. For $k \geq 1$, define $f_k(x)$ to be the reducible polynomial of the form $f_{k-1}(x)+x^n$ with n as small as possible and $n > \deg f_{k-1}$.

$$F(x) = x^n + x^{35} + x^{34} + x^{33} + x^{32} + x^{16} + x^{15} + x^3 + 1$$

2. Let $f_0(x) = 1$. For $k \geq 1$, define $f_k(x)$ to be the reducible polynomial of the form $f_{k-1}(x) + x^n$ with n as small as possible and $n > \deg f_{k-1}$.

$$1$$

$$1 + x^3$$

$$1 + x^3 + x^{15}$$

$$1 + x^3 + x^{15} + x^{16}$$

$$1 + x^3 + x^{15} + x^{16} + x^{32}$$

$$1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33}$$

$$1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34}$$

$$1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34} + x^{35}$$

The list ends with $f_7(x)$.

Why is $x^n + f_7(x)$ irreducible for all $n \geq 36$?

Is the sequence $\{f_k(x)\}$ an infinite sequence?

Why is $x^n + f_7(x)$ irreducible for all $n \geq 36$?

Two Steps:

1. **Handle reciprocal factors (there are none).**
2. **Handle non-reciprocal factors (there is only one).**

Step 1: Handle Reciprocal Factors

Let

$$g(x) = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34} + x^{35}.$$

If f is an irreducible reciprocal factor of

$$F(x) = x^n + g(x),$$

then it divides

$$\widetilde{F}(x) = \tilde{g}(x)x^{n-35} + 1.$$

So f divides

$$\tilde{g}(x)F(x) - x^{35}\widetilde{F}(x) = g(x)\tilde{g}(x) - x^{35}.$$

$$f \text{ divides } g(x)\tilde{g}(x) - x^{35}$$

$$g(x) = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34} + x^{35}$$

$$f \text{ divides } g(x)\tilde{g}(x) - x^{35}$$

Therefore, f is either

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$$

or

$$x^{64} + x^{61} - x^{60} + x^{54} - \cdots - x^{43} + 2x^{42}$$
$$+x^{41} - \cdots + x^{10} - x^4 + x^3 + 1.$$

Recall f divides $F(x) = x^n + g(x)$.

If

$$f = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1,$$

then f also divides $x^7 - 1$.

$$g(x) = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34} + x^{35}$$

$$f \text{ divides } g(x)\tilde{g}(x) - x^{35}$$

$$\text{Recall } f \text{ divides } F(x) = x^n + g(x).$$

If

$$f = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1,$$

then f also divides $x^7 - 1$.

If $n \geq 7$, then f must divide $x^{n-7} + g(x)$.

If $n \geq 14$, then f must divide $x^{n-14} + g(x)$.

If $n \equiv r \pmod 7$, then f must divide $x^r + g(x)$.

Test if $f = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ divides $x^r + g(x)$ for $r \in \{0, 1, 2, 3, 4, 6\}$. It doesn't.

$$g(x) = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34} + x^{35}$$

$$f \text{ divides } g(x)\tilde{g}(x) - x^{35}$$

$$\text{Recall } f \text{ divides } F(x) = x^n + g(x).$$

If

$$f = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1,$$

then f also divides $x^7 - 1$.

Conclusion: The polynomial $F(x) = x^n + g(x)$ is not divisible by $f = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ for any n.

If f is an irreducible reciprocal factor of F, then

$$f(x) = x^{64} + x^{61} - x^{60} + x^{54} - \cdots - x^{43} + 2x^{42} + x^{41} - \cdots + x^{10} - x^4 + x^3 + 1.$$

$$g(x) = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34} + x^{35}$$

$$f(x) = x^{64} + x^{61} - x^{60} + x^{54} - \cdots - x^{43} + 2x^{42} + x^{41} - \cdots + x^{10} - x^4 + x^3 + 1$$

Recall f divides $F(x) = x^n + g(x)$.

Compute the roots of f. In particular, f has a root

$$\alpha \approx 0.58124854 - 0.96349774\ \mathrm{i}$$

with

$$1.125 < |\alpha| < 1.126.$$

$$|g(\alpha)| < g(1.126) < 231 < 1.125^{47} < |\alpha|^{47}$$

$$|F(\alpha)| \geq |\alpha|^n - |g(\alpha)| > 0 \quad \text{for} \quad n \geq 47$$

f does not divide F for any $n \geq 0$

Why is $x^n + f_7(x)$ irreducible for all $n \geq 36$?

Two Steps:

1. Handle reciprocal factors (there are none).
2. Handle non-reciprocal factors (there is only one).

(Maple Time)

Step 2: Handle Non-Reciprocal Factors

$$g(x) = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34} + x^{35}$$

$$F(x) = x^n + g(x)$$

Lemma 2. *Suppose the non-reciprocal part of $F(x) \in \mathbb{Z}[x]$ is reducible, and let $u(x)$ and $v(x)$ be as above. Then the polynomial $w(x) = u(x)\tilde{v}(x)$ has the following properties:*

(i) $w \neq \pm F$ *and* $w \neq \pm \widetilde{F}$.

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Step 2: Handle Non-Reciprocal Factors

$$g(x) = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34} + x^{35}$$

$$F(x) = x^n + g(x)$$

If $n \geq 83$, then

$$F\widetilde{F} = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34} + x^{35} + \cdots$$

where all subsequent terms have degree ≥ 48.

$$w(x) = 1 + \ ??? \ + x^n$$
$$\widetilde{w}(x) = 1 + \ ??? \ + x^n$$

$$w(x) = 1 + x^3 + \cdots + x^n$$
$$\widetilde{w}(x) = 1 + \cdots + x^{n-3} + x^n$$

If $n \geq 83$, then

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$$w(x) = 1 + x^3 + x^{15} + x^{16} + \cdots + x^n$$
$$\widetilde{w}(x) = 1 + \cdots + x^{n-16} + x^{n-15} + x^{n-3} + x^n$$

So $w = F$!!

If $n \geq 83$, then

$$F\widetilde{F} = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34} + x^{35} + \cdots$$

where all subsequent terms have degree ≥ 48.

So $w = F$!!

Lemma 2. ***Suppose the non-reciprocal part of $F(x) \in \mathbb{Z}[x]$ is reducible, and let $u(x)$ and $v(x)$ be as above. Then the polynomial $w(x) = u(x)\tilde{v}(x)$ has the following properties:***

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(v) If F is a $0, 1$-polynomial, then w is also and with the same number of non-zero terms as F.

Why is $x^n + f_7(x)$ irreducible for all $n \geq 36$?

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1. **How does**

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Lemma 2. ***Suppose the non-reciprocal part of*** $F(x) \in \mathbb{Z}[x]$ ***is reducible, and let*** $u(x)$ ***and*** $v(x)$ ***be as above. Then the polynomial*** $w(x) = u(x)\tilde{v}(x)$ ***has the following properties:***

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(Maple Time)