Classifying Reducible Polynomials with Small Norm

Theorem (Schinzel): Fix $a_0, \ldots, a_r \in \mathbb{Z} - \{0\}$. Then there is an algorithm for obtaining a finite classification of the polynomials of the form $a_r x^{d_r} + \cdots + a_1 x^{d_1} + a_0$ that have reducible non-reciprocal part.

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$$\underset{\substack{\uparrow\\ \text{fixed}}}{a_r}\, x^{d_r} + \cdots + \underset{\substack{\uparrow\\ \text{fixed}}}{a_1}\, x^{d_1} + \underset{\substack{\uparrow\\ \text{fixed}}}{a_0}$$

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$$\underset{\text{fixed}}{\underset{\uparrow}{a_r}}\, x^{\overset{\text{variable}}{\overset{\downarrow}{d_r}}} + \cdots + \underset{\text{fixed}}{\underset{\uparrow}{a_1}}\, x^{\overset{\text{variable}}{\overset{\downarrow}{d_1}}} + \underset{\text{fixed}}{\underset{\uparrow}{a_0}}$$

Theorem: If $a > b > c > d > e > 0$, then the non-reciprocal part of

$$f(x) = x^a + x^b + x^c + x^d + x^e + 1$$

is irreducible unless $f(x)$ is a variation of

$$\begin{aligned} f(x) &= x^{5s+3t} + x^{4s+2t} + x^{2s+2t} + x^t + x^s + 1 \\ &= (x^{3s+2t} - x^{s+t} + x^t + 1)(x^{2s+t} + x^s + 1). \end{aligned}$$

Theorem (F. & Murphy): If $n > c > b > a > 0$, then the non-reciprocal part of

$$f(x) = x^n \pm x^c \pm x^b \pm x^a \pm 1$$

is irreducible unless $f(x)$ is a variation of one of the following:

$$x^{8t} - x^{7t} - x^{4t} + x^{2t} - 1 = (x^{3t} - x^t - 1)(x^{5t} - x^{4t} + x^{3t} - x^t + 1)$$

$$x^{8t} - x^{6t} + x^{4t} - x^t - 1 = (x^{3t} - x^{2t} + 1)(x^{5t} + x^{4t} - x^{2t} - x^t - 1)$$

$$x^{9t} - x^{7t} + x^{6t} - x^t - 1 = (x^{3t} - x^{2t} + 1)(x^{6t} + x^{5t} - x^{2t} - x^t - 1)$$

$$x^{10t} - x^{7t} - x^{6t} - x^{4t} - 1 = (x^{3t} - x^t - 1)(x^{7t} + x^{5t} + x^{2t} - x^t + 1)$$

$$x^{10t} - x^{9t} + x^{8t} - x^t - 1 = (x^{3t} - x^{2t} + 1)(x^{7t} + x^{5t} - x^{2t} - x^t - 1)$$

$$x^{10t} - x^{6t} - x^{5t} + x^{4t} - 1 = (x^{5t} - x^{4t} + x^{3t} - x^t + 1)(x^{5t} + x^{4t} - x^{2t} - x^t - 1)$$

$$x^{10t} - x^{9t} - x^{6t} + x^{3t} - 1 = (x^{3t} - x^t - 1)(x^{7t} - x^{6t} + x^{5t} - x^{3t} + x^{2t} - x^t + 1)$$

$$x^{10t} + x^{7t} + x^{4t} - x^t - 1 = (x^{3t} - x^{2t} + 1)(x^{7t} + x^{6t} + x^{5t} + x^{4t} - x^{2t} - x^t - 1)$$

$$x^{11t} - x^{8t} - x^{6t} - x^{5t} - 1 = (x^{4t} - x^{t} + 1)(x^{7t} - x^{3t} - x^{2t} - x^{t} - 1)$$

$$x^{11t} + x^{8t} + x^{6t} - x^{t} - 1 = (x^{3t} - x^{2t} + 1)(x^{8t} + x^{7t} + x^{6t} + x^{5t} - x^{2t} - x^{t} - 1)$$

$$x^{13t} - x^{11t} - x^{9t} - x^{4t} - 1 = (x^{3t} - x^{t} - 1)(x^{10t} + x^{7t} - x^{6t} + x^{5t} + x^{2t} - x^{t} + 1)$$

$$x^{13t} - x^{11t} + x^{10t} - x^{2t} - 1 = (x^{5t} - x^{4t} + x^{2t} - x^{t} + 1)(x^{8t} + x^{7t} - x^{2t} - x^{t} - 1)$$

$$\begin{aligned} x^{14t} - x^{11t} + x^{9t} - x^{3t} - 1 &= (x^{7t} - x^{6t} + x^{3t} - x^{t} + 1) \\ &\quad \times (x^{7t} + x^{6t} + x^{5t} - x^{3t} - x^{2t} - x^{t} - 1) \end{aligned}$$

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$$x^{2t+u} - x^{t+2u} + x^{2u} - x^{t} - 1 = (x^{t} - x^{u} + 1)(x^{t+u} - x^{u} - 1)$$

$$x^{5t+2u} - x^{4t+2u} - x^{t+u} - x^{t} - 1 = (x^{2t+u} - x^{t+u} - 1)(x^{3t+u} + x^{t} + 1)$$

$$x^{5t+3u} - x^{4t+2u} - x^{t+u} - x^{t} - 1 = (x^{2t+u} - x^{t} - 1)(x^{3t+2u} + x^{t+u} + 1)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Lemma: Let s and t be positive integers. Consider a system of linear equations in the variables $x_0, \ldots, x_s$ of the form

$$\alpha_{i0}x_0 + \alpha_{i1}x_1 + \cdots + \alpha_{is}x_s = \beta_i \quad (1 \leq i \leq t),$$

where the α_{ij} and β_i are all in $\mathbb{Z}$.

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$$\begin{pmatrix} \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,s} \\ \alpha_{2,0} & \alpha_{2,1} & \cdots & \alpha_{2,s} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{t,0} & \alpha_{t,1} & \cdots & \alpha_{t,s} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_s \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_t \end{pmatrix}$$

infinitely many solutions

one with $x_j \in \mathbb{Z}$ distinct $\implies$ infinitely many

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$$A = (\alpha_{i,j-1})_{t\times(s+1)},$$

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$$A = (\alpha_{i,j-1})_{t\times(s+1)}, \quad \rho = \text{rank of } A$$

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$$A = (\alpha_{i,j-1})_{t\times(s+1)}, \quad \rho = \text{rank of } A$$

rearrange so first ρ rows are linearly independent

$$\begin{pmatrix} \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,s} \\ \alpha_{2,0} & \alpha_{2,1} & \cdots & \alpha_{2,s} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{t,0} & \alpha_{t,1} & \cdots & \alpha_{t,s} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_s \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_t \end{pmatrix}$$

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rearrange so first ρ rows are linearly independent

rearrange so first ρ columns are linearly independent

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$A = (\alpha_{i,j-1})_{t\times(s+1)}$, $\quad \rho =$ rank of A

rearrange so first ρ rows are linearly independent

rearrange so first ρ columns are linearly independent

$$B = \begin{pmatrix} \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,\rho-1} \\ \alpha_{2,0} & \alpha_{2,1} & \cdots & \alpha_{2,\rho-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{\rho,0} & \alpha_{\rho,1} & \cdots & \alpha_{\rho,\rho-1} \end{pmatrix} = (\alpha_{i,j-1})_{\rho\times\rho}$$

$$\begin{pmatrix} \multicolumn{3}{c}{\multirow{2}{*}{$\boldsymbol{B}$}} & \boldsymbol{\alpha_{1,\rho}} & \cdots & \boldsymbol{\alpha_{1,s}} \\ & & & \boldsymbol{\alpha_{2,\rho}} & \cdots & \boldsymbol{\alpha_{2,s}} \\ & & & \vdots & \ddots & \vdots \\ \boldsymbol{\alpha_{\rho+1,0}} & \cdots & & \boldsymbol{\alpha_{\rho+1,\rho}} & \cdots & \boldsymbol{\alpha_{\rho+1,s}} \\ \vdots & & & \vdots & \ddots & \vdots \\ \boldsymbol{\alpha_{t,0}} & \cdots & & \boldsymbol{\alpha_{t,1}} & \cdots & \boldsymbol{\alpha_{t,s}} \end{pmatrix} \begin{pmatrix} \boldsymbol{x_0} \\ \boldsymbol{x_1} \\ \vdots \\ \boldsymbol{x_s} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\beta_1} \\ \boldsymbol{\beta_2} \\ \vdots \\ \boldsymbol{\beta_t} \end{pmatrix}$$

$$\begin{pmatrix} \multicolumn{3}{c}{\multirow{2}{*}{$\boldsymbol{B}$}} & \alpha_{1,\rho} & \cdots & \alpha_{1,s} \\ & & & \alpha_{2,\rho} & \cdots & \alpha_{2,s} \\ & & & \vdots & \ddots & \vdots \\ \alpha_{\rho+1,0} & \cdots & & \alpha_{\rho+1,\rho} & \cdots & \alpha_{\rho+1,s} \\ \vdots & & & \vdots & \ddots & \vdots \\ \alpha_{t,0} & \cdots & & \alpha_{t,1} & \cdots & \alpha_{t,s} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_s \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_t \end{pmatrix}$$

$$B \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{\rho-1} \end{pmatrix} = \begin{pmatrix} \beta_1 - \alpha_{1,\rho} x_\rho - \cdots - \alpha_{1,s} x_s \\ \beta_2 - \alpha_{2,\rho} x_\rho - \cdots - \alpha_{2,s} x_s \\ \vdots \\ \beta_\rho - \alpha_{\rho,\rho} x_\rho - \cdots - \alpha_{\rho,s} x_s \end{pmatrix}$$

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$$x_i = \frac{1}{D}\Big(c_i + \sum_{j=\rho}^{s} b_{ij} x_j\Big) \qquad \begin{matrix} 0 \leq i \leq \rho - 1 \\ c_i, b_{ij} \in \mathbb{Z},\, D = |\det B| \end{matrix}$$

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Fix a solution $(k_0, k_1, \ldots, k_s)$ with $k_j \in \mathbb{Z}$ distinct.

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Want other solutions $(k'_0, k'_1, \ldots, k'_s)$ with k'_j distinct.

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Define

$$k'_i = k_i + \ell_i D \quad \text{for} \quad \rho \le i \le s \quad (\ell_i \in \mathbb{Z}, \text{ large})$$

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Then $(k'_0, k'_1, \ldots, k'_s)$ is a solution in distinct integers.

$$x_i = \frac{1}{D}\Big(c_i + \sum_{j=\rho}^{s} b_{ij} x_j\Big) \qquad \begin{array}{c} 0 \le i \le \rho - 1 \\ c_i, b_{ij} \in \mathbb{Z}, D = |\det B| \end{array}$$

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Then $(k'_0, k'_1, \ldots, k'_s)$ is a solution in $\overset{?}{\text{distinct}}$ integers.

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Fix a solution $(k_0, k_1, \ldots, k_s)$ with $k_j \in \mathbb{Z}$ distinct.

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Define

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? ?

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For $\rho \le i \le s$, define $k'_i = k_i + \ell_i D$.

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For $0 \le i \le \rho - 1$, define

$$k'_i = \frac{1}{D}\Big(c_i + \sum_{j=\rho}^{s} b_{ij}k_j\Big) + \sum_{j=\rho}^{s} b_{ij}\ell_j$$

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For $\rho \le i \le s$, define $k'_i = k_i + \ell_i D$.

For $0 \le i \le \rho - 1$, define

$$k'_i = \qquad k_i \qquad + \sum_{j=\rho}^{s} b_{ij}\ell_j$$

$$x_i = \frac{1}{D}\Big(c_i + \sum_{j=\rho}^{s} b_{ij} x_j\Big) \qquad \begin{array}{c} 0 \le i \le \rho - 1 \\ c_i, b_{ij} \in \mathbb{Z},\ D = |\det B| \end{array}$$

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Observe that the k'_j's are integers (if the ℓ_j's are).

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Hence, the k'_j's are distinct.

Lemma: Let s and t be positive integers. Consider a system of linear equations in the variables $x_0, \ldots, x_s$ of the form

$$\alpha_{i0}x_0 + \alpha_{i1}x_1 + \cdots + \alpha_{is}x_s = \beta_i \quad (1 \leq i \leq t),$$

where the α_{ij} and β_i are all in $\mathbb{Z}$. Suppose the system of equations has infinitely many solutions $(x_0, \ldots, x_s) \in \mathbb{R}^{s+1}$. If the system has at least one solution in $\mathbb{Z}^{s+1}$ with x_0, $x_1, \ldots,$ x_s *distinct*, then the system has infinitely many such solutions.

Theorem (Schinzel): Fix $a_0, \ldots, a_r \in \mathbb{Z} - \{0\}$. Then there is an algorithm for obtaining a finite classification of the polynomials of the form $a_r x^{d_r} + \cdots + a_1 x^{d_1} + a_0$ that have reducible non-reciprocal part.

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How can we determine if its
non-reciprocal part is irreducible?

$$f(x) = \sum_{j=0}^{r} a_j x^{d_j} \quad \text{and} \quad w(x) = \sum_{j=0}^{s} b_j x^{k_j}$$

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b_j, k_j are unknown integers

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Fix the b_j's. Solve for the k_j's.

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Equate elements from E with the exponents in $f\tilde{f}$.
One obtains various systems of equations.
Solve them.

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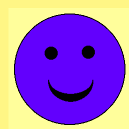
(i′) There's a unique solution. 

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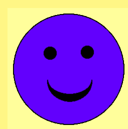
(i′) There's a unique solution. 

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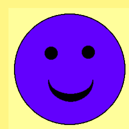
(i′) There's a unique solution. 

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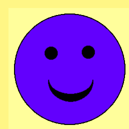
(i$'$) There's a unique solution. 

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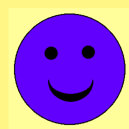
(i′) There's a unique solution. 

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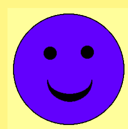
(i$'$) There's a unique solution. 

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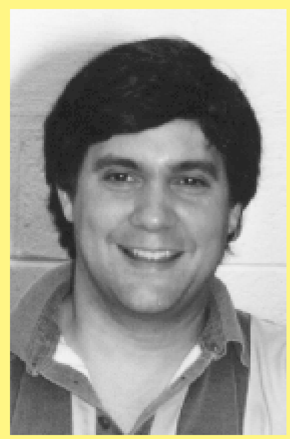

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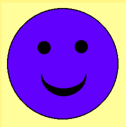
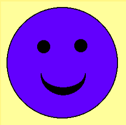
(i′) There's a unique solution. 

(ii′) There's no solution. 

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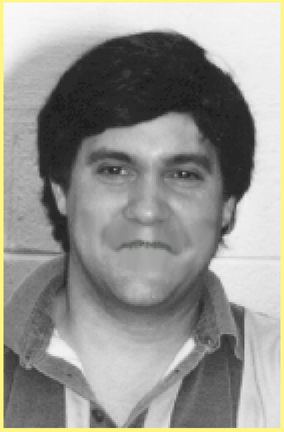

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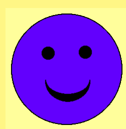
(i$'$) There's a unique solution. 

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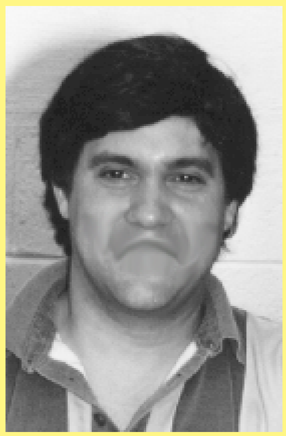

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$$k'_0 = 0 \text{ and } k'_s = n \implies k'_u \le 0 \text{ and } k'_v \ge n.$$

Hence,

$$n - k'_v + k'_u \le -1.$$

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or

$$n - k'_v + k'_u = m$$

for some exponent m appearing in $f(x)\tilde{f}(x)$.

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Either

$$n - k'_v + k'_u < n - k'_j + k'_i, \quad (i,j) \neq (u,v)$$

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solution in distinct integers $k_0', k_1', \ldots, k_s'$

$$n - k_v' + k_u' \leq -1$$

Either

$$n - k_v' + k_u' = n - k_j' + k_i', \quad (i, j) \neq (u, v)$$

or

$$n - k_v' + k_u' = m$$

for some exponent m appearing in $f(x)\tilde{f}(x)$.

solution in distinct integers $k'_0, k'_1, \ldots, k'_s$

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We have a contradiction, and the claim follows.

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- This gives a classification of f for which w exists.
- Solve more systems to see when w is $\pm f$ or $\pm\tilde{f}$.