MATH 788F: PRACTICE TEST

(1) Prove the following:

> Let $f(x)$ be a monic polynomial in $\mathbb{Z}[x]$ for which $f(0) \neq 0$. Suppose further that $f(x) = f_1(x)f_2(x)f_3(x)$ where each $f_j(x)$ is irreducible. If
> $$f(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$$
> where each $\alpha_j \in \mathbb{C}$, then $|\alpha_j| \geq 1$ for at least three values of $j \in \{1, 2, \dots, n\}$.

(2) Let $f(x) = x^3 + 22$. Determine with proof all primes p for which $f(x)$ is Eisenstein with respect to p. For each such p, find a value of a for which $f(x + a)$ is in Eisenstein form with respect to p.

(3) Let
$$f(x) = x^7 + 21x^6 - 30x^4 - 90x^3 + 1350x + 2700.$$
Using Newton polygons, explain why $f(x)$ is irreducible. (Be careful, and indicate as clearly as possible what information you are obtaining from each Newton polygon you use in your argument.)

(4) Let
$$f(x) = x^{16} - 8x^{15} - 4x^{14} - 2x^{13} - x^{12} - x^{11} - x^{10} - \cdots - x - 1.$$
For each part below, give all details of your solution. Do NOT refer to any theorems from class (in particular, do not refer to Perron's Theorem or its proof). You may however use the following lemmas from class:

Lemma 1. Let $f(x)$ be a monic polynomial in $\mathbb{Z}[x]$ for which $f(0) \neq 0$. Suppose further that $f(x)$ has exactly 1 root α (with multiplicity 1) such that $|\alpha| \geq 1$. Then $f(x)$ is irreducible.

Lemma 2. Let $f(x)$ and $g(x)$ be polynomials in $\mathbb{C}[x]$, and let $\mathcal{C} = \{z \in \mathbb{C} : |z| = 1\}$. If the strict inequality $|f(z) + g(z)| < |f(z)| + |g(z)|$ holds for each $z \in \mathcal{C}$, then $f(x)$ and $g(x)$ have the same total number of zeroes (counting multiplicity) inside the circle $\mathcal{C}$ (i.e., in the interior of the region bounded by $\mathcal{C}$).

(a) Consider $F(x) = (2x - 1)f(x)$. Explain why $F(x)$ has exactly one root α satisfying $|\alpha| \geq 1$. (Hint: Expand the product $(2x - 1)f(x)$.)

(b) Explain why this implies that $f(x)$ is irreducible.

(5) For p a prime, prove that the Bernoulli polynomial $B_{(2p-1)(p-1)}(x)$ is irreducible.