MATH 788F

Practice Test Problems

(1) Prove the following:

Let $f(x)$ be a monic polynomial in $\mathbb{Z}[x]$ for which $f(0) \neq 0$. Suppose further that $f(x) = f_1(x)f_2(x)f_3(x)$ where each $f_j(x)$ is irreducible. If

$$f(x) = (x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n)$$

where each $\alpha_j \in \mathbb{C}$, then $|\alpha_j| \geq 1$ for at least three values of $j \in \{1, 2, \ldots, n\}$.

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> where each $\alpha_j \in \mathbb{C}$, then $|\alpha_j| \geq 1$ for at least three values of $j \in \{1, 2, \dots, n\}$.

Show each $f_j(x)$ has a root with absolute value ≥ 1.

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$$|f_j(0)| = |\alpha_1'|\,|\alpha_2'| \cdots |\alpha_r'|$$

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$$\implies \text{there exists } j \text{ such that } |\alpha'_j| \geq 1$$

(4) Let

$$\begin{aligned} f(x) = x^{16} - 8x^{15} - 4x^{14} - 2x^{13} \\ - x^{12} - x^{11} - x^{10} - \cdots - x - 1. \end{aligned}$$

(a) Consider $F(x) = (2x - 1)f(x)$. Explain why $F(x)$ has exactly one root α satisfying $|\alpha| \geq 1$.

(b) Explain why this implies that $f(x)$ is irreducible.

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Lemma 2. Let $f(x)$ and $g(x)$ be polynomials in $\mathbb{C}[x]$, and let $\mathcal{C} = \{z \in \mathbb{C} : |z| = 1\}$. If the strict inequality $|f(z) + g(z)| < |f(z)| + |g(z)|$ holds for each $z \in \mathcal{C}$, then $f(x)$ and $g(x)$ have the same total number of zeroes (counting multiplicity) inside the circle $\mathcal{C}$ (i.e., in the interior of the region bounded by $\mathcal{C}$).

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Idea: Take $m = (2p-1)(p-1)$, and show that

$$p\widetilde{B}_m(x) = \sum_{j=0}^{m} pB_j \binom{m}{j} x^j$$

is a rational number times a polynomial in Eisenstein form with respect to p.

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Use that $(p-1)|m$ implies B_m is a rational number which when reduced has denominator divisible by p but not by p^2.

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So p divides the constant term and p^2 does not.

$$m = (2p-1)(p-1), \quad f(x) = \sum_{j=0}^{m} pB_j \binom{m}{j} x^j$$

Other Terms where $(p-1) \nmid j$:

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Other Terms where $(p-1) \nmid j$:

Use that B_j is a rational number which when reduced has both numerator and denominator *not* divisible by p.

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Panic and hope for partial credit or use the lemma on the power of p dividing a factorial.

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Other Terms where $(p-1)|j$:

$$m = 2p^2 - 3p + 1$$

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$s(m) = p-1$

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$$s(m) = p - 1 \implies \nu_p(m!) = \frac{m - s(m)}{p-1}$$

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Other Terms where $(p-1)|j$:

$$m = 2p^2 - 3p + 1 = p^2 + (p-3)p + 1$$

$$s(m) = p-1 \implies \nu_p(m!) = \frac{m - s(m)}{p-1} = 2p-2$$

$$m = (2p-1)(p-1), \quad f(x) = \sum_{j=0}^{m} pB_j \binom{m}{j} x^j$$

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$$\nu_p\left(\binom{m}{j}\right) = \nu_p(m!) - \nu_p(j!) - \nu_p\big((m-j)!\big)$$

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$$\begin{aligned} \nu_p\left(\binom{m}{j}\right) &= \nu_p(m!) - \nu_p(j!) - \nu_p\big((m-j)!\big) \\ &\geq 2p - 2 - \frac{j}{p-1} - \frac{m-j}{p-1} + 2 \end{aligned}$$

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