MATH 784: ALGEBRAIC NUMBER THEORY

COURSE DESCRIPTION:

The purpose of this course is to describe methods in which algebra can be used as a tool to determine information about the integers. The algebraic concepts used during this course will be introduced in the course, so that a great deal of background is not necessary to enroll in this class. In particular, it will not be assumed that students have taken other courses in graduate number theory. For example, the first day will be spent introducing the concept of algebraic integers and then we will use this notion to determine all rational numbers θ such that $\sin^2(\pi\theta)$ is rational. Some other questions and problems we will discuss include:

(1) Find the smallest postive integer x such that for some positive integer y, $x^2 - 89y^2 = 1$.

(2) What primes can be written as a sum of two squares?

(3) Is the inequality

$$\left[\frac{\pi}{2}n\right] < \frac{\pi}{2}n^2 \sin\left(\frac{1}{n}\right) \leq \frac{\pi}{2}n,$$

where $[x]$ denotes the greatest integer $\leq x$, true for all positive integers n?

(4) Show that if $y^2 + 4 = x^3$ where x and y are positive integers, then $(x, y) = (2, 2)$ or $(5, 11)$.

(5) How can we find primes with more than $50,000$ digits and why is $2^{216091} - 1$ prime?

(6) Show that if $a_1, a_2, \ldots, a_{n-1}$ are arbitrary integers, then

$$\frac{x^n}{n!} + a_{n-1}\frac{x^{n-1}}{(n-1)!} + \cdots + a_2\frac{x^2}{2!} + a_1 x \pm 1$$

is irreducible over the rationals.

(7) Show that if x is a positive integer for which $x^2 + 7$ is a power of 2, then x must be 1, 3, 5, 11, or 181.