MATH 783: ANALYTIC NUMBER THEORY II

COURSE DESCRIPTION:

The purpose of this course is to expose students to the method of exponential sums. Specifically, the method produces upper bounds for the absolute value of sums of the form

$$\sum_{N<n\le 2N} e^{2\pi i f(n)},$$

where $f(n)$ is a "nice" function. Most of the course is centered around making such estimates. These methods are used to obtain number theoretic results. Some of the number theoretic results we hope to discuss are the following:

(1) If α is an irrational number, then the sequence $\{n\alpha\}$ is uniformly distributed modulo 1. For example, this would mean that the fractional part of $n\alpha$ (the part of the number which occurs after the decimal) occurs in the interval $(1/4, 1/2]$ with probability $1/4$.

(2) A squarefree number is an integer which is not divisible by p^2 for every prime p. Thus, $6, 7$, and 15 are squarefree and $8, 12$, and 18 are not. We will show that for x sufficiently large, every interval of the form $(x, x + x^{2/9}]$ contains a squarefree number.

(3) The Goldbach Conjecture is that every even number which is ≥ 4 is a sum of 2 primes. For example, $4 = 2 + 2$, $6 = 3 + 3$, and $8 = 3 + 5$. This is still an open problem; however, using an exponential sum approach, it is possible to show that almost every even number can be written as a sum of 2 primes. More specifically, the quantity

$$\frac{|\{n \le x : n \text{ is even, } n \text{ is the sum of 2 primes}\}|}{|\{n \le x : n \text{ is even}\}|}$$

tends to 1 as x approaches infinity.

The method also has applications to the Riemann Zeta Function and to Dirichlet's Divisor Problem (which will be described in the course).