MATH 241: FINAL EXAM

Name _____

Instructions and Point Values: Put your name in the space provided above. Check that your test contains 14 different pages including one blank page. Work each problem below and show ALL of your work. Unless stated otherwise, you do not need to simplify your answers. Do NOT use a calculator.

There are 300 total points possible on this exam. The points for each problem in each part is indicated below.

PART I

Problem (1) is worth 20 points.

Problem (2) is worth 15 points.

Problem (3) is worth 15 points.

Problem (4) is worth 18 points.

Problem (5) is worth 18 points.

Problem (6) is worth 24 points.

Problem (7) is worth 20 points.

Problem (8) is worth 20 points.

PART II

Problem (1) is worth 30 points.

Problem (2) is worth 40 points.

Problem (3) is worth 40 points.

Problem (4) is worth 40 points.

PART I. Answer each of the following.

(1) Let $\vec{u} = \langle 2, -1, -2 \rangle$ and $\vec{v} = \langle 3, 1, -1 \rangle$. Calculate:

(a) $\vec{u} - 2\vec{v}$

(b) the dot product of $\vec{u}$ and $\vec{v}$

(c) $\vec{u} \times \vec{v}$

(d) the length (or magnitude) of $\vec{u}$

(2) Find the volume of the pyramid with vertices $P = (-1, 1, -1)$, $Q = (2, 1, 1)$, $R = (-1, 2, 0)$, and $S = (-1, 1, 0)$.

(3) Let $\vec{v} = \langle -1, 1 \rangle$ and $f(x, y) = x^2 + xy$. Calculate the directional derivative of $f(x, y)$ in the direction $\vec{v}$ at the point $(-1, 2)$.

(4) Calculate cylindrical coordinates (r, θ, z) and spherical coordinates (ρ, ϕ, θ) for the point with rectangular coordinates $(x, y, z) = (3, 3\sqrt{3}, -2\sqrt{3})$.

(r, θ, z):

(ρ, ϕ, θ):

(5) Find an equation for the tangent plane to the surface $(x + y)(x + y + z) = 2z^2$ at the point $(3, -1, 2)$.

(6) Calculate the following integrals. **SIMPLIFY** your answers.

(a) $\displaystyle\int_0^1 \int_0^x (x-y)\,dy\,dx$

(b) $\displaystyle\int_0^\pi \int_0^\theta \int_0^\phi d\rho\,d\phi\,d\theta$

(7) Let $\vec{F} = x^2\,\mathbf{i} + xy\,\mathbf{j} + y^2\,\mathbf{k}$. Calculate the divergence and curl of $\vec{F}$.

Divergence: ____

Curl: ____

(8) Let

$$f(x, y) = x^2y^2 - x^2 + y^2 - 4y + 2.$$

The function $f(x, y)$ has 3 critical points. Calculate the critical points and indicate (with justification) whether each determines a local maximum value of $f(x, y)$, a local minimum value of $f(x, y)$, or a saddle point of $f(x, y)$.

1) FIRST CRITICAL POINT: [ ]

LOCAL MAX, LOCAL MIN, OR SADDLE PT: [ ]

2) SECOND CRITICAL POINT: [ ]

LOCAL MAX, LOCAL MIN, OR SADDLE PT: [ ]

3) THIRD CRITICAL POINT: [ ]

LOCAL MAX, LOCAL MIN, OR SADDLE PT: [ ]

PART II. Answer each of the following. Make sure your work is clear. If you do not know how to answer a problem, tell me what you know that you think is relevant to the problem. If you end up with an answer that you think is incorrect, tell me this as well. Better yet, tell me why you think it is incorrect. In other words, let me know what you know.

(1) The graphs for the equations below are similar to 2 graphs on the last page of this test. The orientation and the scaling may be different. For each equation, indicate which graph on the last page best matches it. For example, if the equation is for a hyperbolic paraboloid, then the graph you choose should be a hyperbolic paraboloid. Indicate your choice by putting the corresponding letter from the last page in the box under the equation below. Next, read the question on the last page corresponding to the graph you chose. Then go back to the equation below and put the answer to the question for the graph of that equation in the box. Do **NOT** answer the question for the graph on the last page (since it may be oriented differently than the graph of the equation below).

(a) $z^2 + 2x^2 = 3y^2$

ANSWER: [ ]

(b) $z^2 + 2x^2 = 3y^2 - 1$

ANSWER: [ ]

(2) Calculate the following integrals. (They are in the second part of this exam for a reason.)

(a) $\displaystyle\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} \left(x^2+y^2\right)^{1/2}\, dx\, dy$

(b) $\displaystyle\int_{-1}^{1}\int_{x^2}^{1} \sin(y^{3/2})\, dy\, dx$

(c) $\displaystyle\int_{-4}^{4}\int_{0}^{\sqrt{16-x^2}}\int_{0}^{\sqrt{16-x^2-y^2}}\sqrt{1+(x^2+y^2+z^2)^{3/2}}\,dz\,dy\,dx$

(3) Let $\mathcal{P}$ be the plane $3x + y - 4z = 7$, let ℓ_1 be the line given by $x = 1 + 2t$, $y = 1 - 2t$, and $z = 3 + t$, and let ℓ_2 be the line given by $x = -1 + t$, $y = 2 + t$, and $z = -2 + t$.

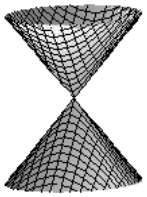
(a) Explain why ℓ_1 does not intersect the plane $\mathcal{P}$.

(b) Does ℓ_2 intersect the plane $\mathcal{P}$? Justify your answer.

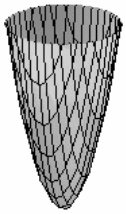
(c) Calculate the minimum distance from ℓ_1 to ℓ_2. (You should justify your work; in particular, if you happen to have memorized a formula for such a distance, you should prove that the formula works.)

(4) Find the points on the graph of $z = x^2 - 3y^2 - 1$ which are closest to the origin $(0, 0, 0)$. Justify your answer.

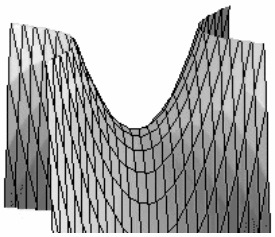
(a) This is a graph of an elliptic cone. Planes parallel to the $xy-$plane intersect this graph in ellipses (except for the plane $z = 0$ which gives only a point). The equation on page 7 corresponding to this graph also produces ellipses when intersected by the right planes. What are the right planes for your graph? In other words, what are they parallel to (the xy-plane, the xz-plane, or the yz-plane)?



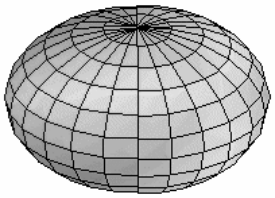
(b) This is a graph of an elliptic paraboloid. What are the coordinates of the vertex for the corresponding equation on page 7?



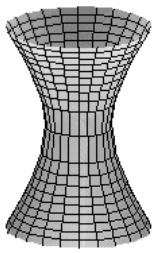
(c) This is a hperbolic paraboloid. There is a plane $z = k$ which intersects the graph for the corresponding equation on page 7 in 2 lines. What is the value of k and what are the equations of the 2 lines?



(d) This is an ellipsoid. What is the center of the ellipsoid for the graph of the corresponding equation on page 7.



(e) This is a graph of a hyperboloid of 1 sheet. There is a plane $z = k$ which intersects the graph for the corresponding equation on page 7 in 2 lines. What is the value of k and what are the equations of the 2 lines? (There are two correct values of k for this problem.)



(f) This is a graph of a hyperboloid of 2 sheets. The vertices of the hyperboloid of 2 sheets drawn here are the 2 points where the surface intersects the $z-$axis. What are the vertices for the corresponding graph on page 7? (Give the three coordinates for each vertex.)