Math 241: Calculus III Name ______________________

Final Exam (300 points total)

Show All Work

This test consists of 2 parts. There are 10 problems in Part I, each worth 18 points; and there are 5 problems in Part II, each worth 24 points. An asterisk (*) next to a problem indicates that no work is required. Good Luck!!

Part I

(1) Let $\mathbf{a} =< 2, 6, 9 >$ and $\mathbf{b} =< -2, 9, -6 >$. Calculate each of the following, and simplify your answers.

(a) $|\mathbf{a}|$ (the magnitude of $\mathbf{a}$)

(b)* $\mathbf{a} \cdot \mathbf{b}$

(2) Calculate cylindrical coordinates (r, θ, z) and spherical coordinates (ρ, ϕ, θ) for the point with rectangular coordinates $(x, y, z) = (-2, -\sqrt{12}, 4)$.

(r, θ, z): $\boxed{}$

(ρ, ϕ, θ): $\boxed{}$

(3) Calculate the following multiple integrals.

(a) $\int_0^{\pi} \int_0^{\pi - x} dy dx$

(b) $\int_0^{\pi} \int_0^{\pi - x} \frac{\sin y}{\pi - y} dy dx$

(4) Calculate the area enclosed by the graph of the polar equation $r = 2 + \cos\theta$.

(5) Calculate $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x+y^2}{\sqrt{x^2+y^2}}$ or explain why it does not exist.

(6) The function $f(x,y) = x^2 + 2y^2 - 2xy - 4y + 3$ is defined for all points (x,y) in the plane. Explain why $f(x,y)$ has an absolute maximum value or an absolute minimum value and calculate that value. Justify your answer.

(7) Let $f(x, y, z) = x^2 + 2y^2z + y^2 - 2$. Note that the point $P = (1, 1, 0)$ is a point on the graph of $f(x, y, z) = 0$. Calculate the directional derivative of $f(x, y, z)$ at the point P in the direction of the vector $< 6, -3, -2 >$. Simplify your answer.

(8) (a)* In problem (7) above, what is the maximum value of the directional derivative for $f(x, y, z)$ at the point P?

(b)* In problem (7) above, what is the equation of the tangent plane to the surface $f(x, y, z) = 0$ at P?

(9)* The graphs for the equations below are similar to 2 graphs on the last 2 pages of this test. The orientation and the scaling may be different. For each equation, indicate which graph on the last 2 pages best matches it. For example, if the equation is for a hyperbolic paraboloid, then the graph you choose should be a hyperbolic paraboloid. Indicate your choice by putting the corresponding letter from the last 2 pages in the box under the equation below. Next, read the question on the last 2 pages corresponding to the graph you chose. Then go back to the equation below and put the answer to the question for the graph of that equation in the box. Do **NOT** answer the question for the graph on the last 2 pages (since it may be oriented differently than the graph of the equation below).

(a) $2x^2 + z^2 = 3y^2 - 1$

ANSWER: $\boxed{}$

(b) $2x^2 + z^2 = 3y - 1$

ANSWER: $\boxed{}$

(10) Calculate the line integral $\int_{\mathcal{C}} (4x - 3y)\, ds$ where $\mathcal{C}$ is the line segment from $(1, 2)$ to $(4, 6)$. Simplify your answer.

Part II

(1) Let $\mathbf{a} =< 7, 4, 0 >$, $\mathbf{b} =< -6, -4, 1 >$, $\mathbf{c} = \mathbf{a} \times \mathbf{b}$. Viewing these three vectors as emanating from the origin, they form three edges of a parallelepiped. Calculate the volume of the parallelepiped. Simplify your answer.

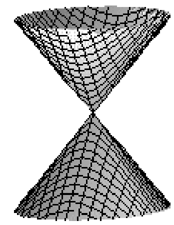
(2) Using the second derivative test for functions of two variables, find all points (a, b, c) where the graph of $f(x, y) = 3y^4 - 5y^2 + 2xy + x^2 + 3$ has a local maximum, a local minimum, or a saddle point. For each such point, indicate which (a local maximum, a local minimum, or a saddle point) occurs.

(3) Use Green's Theorem to calculate $\int_{\mathcal{C}} (y + \sin x)\,dx + (3x - y^3 \cos y)\,dy$, where $\mathcal{C}$ is the counter-clockwise oriented curve consisting of the line segment from $(0,0)$ to $(2,2)$, the portion of the circle $x^2 + y^2 = 8$ from $(2,2)$ to $(-2,-2)$, and the line segment from $(-2,-2)$ to $(0,0)$.

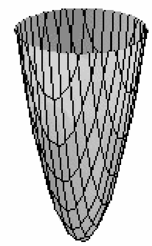
(4) Calculate $\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{\sqrt{2x^2+2y^2}}^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz dy dx$. Simplify so that your final answer does not involve trignometric functions.

(5) Let ℓ_1 be the line with parametric equations $x = 1 - 4t$, $y = 1 - t$, and $z = -1 + t$, and let ℓ_2 be the line with parametric equations $x = 3$, $y = 1 - 2t$, and $z = 1 + t$. Then ℓ_1 and ℓ_2 are two lines in space which do not intersect. Calculate the distance between these lines. Simplify your answer. (The distance between ℓ_1 and ℓ_2 is the minimum value of $\left|\overrightarrow{PQ}\right|$ where P is a point on ℓ_1 and Q is a point on ℓ_2. This problem can be done using a method similar to the approach we used to calculate the distance between two planes.)

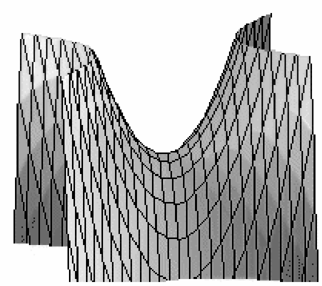
(a) This is a graph of an elliptic cone. Planes parallel to the $xy-$plane intersect this graph in ellipses (except for the plane $z = 0$ which gives only a point). The equation on page 5 corresponding to this graph also produces ellipses when intersected by the right planes. What are the right planes for your graph? In other words, what are they parallel to (one of the xy-plane, the xz-plane, or the yz-plane)?



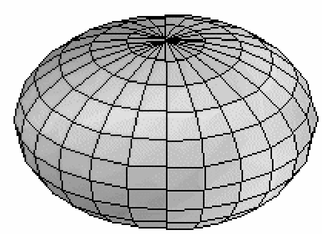
(b) This is a graph of an elliptic paraboloid. What are the coordinates of the vertex for the corresponding equation on page 5?



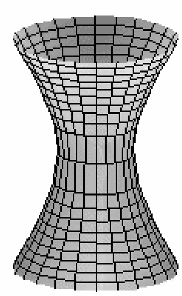
(c) This is a hperbolic paraboloid. There is a plane $z = k$ which intersects the graph for the corresponding equation on page 5 in 2 lines. What is the value of k and what are the equations of the 2 lines?



(d) This is an ellipsoid. What is the center of the ellipsoid for the graph of the corresponding equation on page 5.



(e) This is a graph of a hyperboloid of 1 sheet. There is a plane $z = k$ which intersects the graph for the corresponding equation on page 5 in 2 lines. What is the value of k and what are the equations of the 2 lines? (There are two correct values of k for this problem.)



(f) This is a graph of a hyperboloid of 2 sheets. The vertices of the hyperboloid of 2 sheets drawn here are the 2 points where the surface intersects the $z-$axis. What are the vertices for the corresponding graph on page 5? (Give the three coordinates for each vertex.)