Lagrange Multiplier Problem From 1998 Final Exam

Part II, Problem 2: The maximum and minimum values must occur at points (x, y) where

$$6xy + 9x^2 = \lambda 2x, \quad 3x^2 + 6y^2 = \lambda 2y, \quad \text{and} \quad x^2 + y^2 = 1,$$

for some λ.

First, suppose $x = 0$, so the first equation is satisfied. Since $x = 0$ and $x^2 + y^2 = 1$, we must have $y = \pm 1$, and the third equation is satisfied. Substituting $x = 0$ and $y = \pm 1$ into the second equation, we get $6 = \pm 2\lambda$. Taking $\lambda = \pm 3$, we see that the second equation is also satisfied. So we need to consider the two points $(x, y) = (0, \pm 1)$.

Now, suppose $x \neq 0$. Dividing by x in the first equation, we obtain $6y + 9x = 2\lambda$. Substituting this value for 2λ in the second equation gives $3x^2 + 6y^2 = (6y + 9x)y = 6y^2 + 9xy$. Subtracting $6y^2$ from both sides, we get $3x^2 = 9xy$. Since $x \neq 0$, we can divide by $3x$ to get $x = 3y$. Plugging this into the constraint $x^2 + y^2 = 1$, we obtain $10y^2 = 1$ so $y = \pm 1/\sqrt{10}$ so that $x = \pm 3/\sqrt{10}$ (with x and y of the same sign since $x = 3y$). Thus, we need to consider the two points $(3/\sqrt{10}, 1/\sqrt{10})$ and $(-3/\sqrt{10}, -1/\sqrt{10})$.

Calculating $f(x, y)$ for each of the points

$$(0, \pm 1), \quad (3/\sqrt{10}, 1/\sqrt{10}), \quad \text{and} \quad (-3/\sqrt{10}, -1/\sqrt{10})$$

gives that the maximum value of $f(x, y)$ given the constraint $x^2 + y^2 = 1$ is $f(3/\sqrt{10}, 1/\sqrt{10}) = 11/\sqrt{10}$ and the minimum value is $f(-3/\sqrt{10}, -1/\sqrt{10}) = -11/\sqrt{10}$.
