Old Math 241 Test 3's

Some 1992 Solutions:

1. (a) $\displaystyle\int_0^1\int_y^{y^2} y\,dx\,dy = \int_0^1 yx\Big|_{x=y}^{y^2} dy = \int_0^1 (y^3-y^2)\,dy = \frac{y^4}{4}-\frac{y^3}{3}\Big|_0^1 = \frac{1}{4}-\frac{1}{3} = \frac{-1}{12}$

   (b) $\displaystyle\int_0^\pi\int_0^2 \sin\theta\,dr\,d\theta = \int_0^\pi (\sin\theta)\,r\Big|_{r=0}^2 d\theta = \int_0^\pi 2\sin\theta\,d\theta = -2\cos\theta\Big|_0^\pi = 2+2=4$

   (c) The region of integration is above the x-axis, to the right of $y=x^2$ and to the left of $x=1$. Changing the order of integration gives

   $$\int_0^1\int_0^{x^2} e^{x^3}\,dy\,dx = \int_0^1 e^{x^3}y\Big|_{y=0}^{x^2} dx = \int_0^1 x^2e^{x^3}\,dx = \frac{e^{x^3}}{3}\Big|_0^1 = \frac{e-1}{3}.$$

   Note that the last integral can be done by the u-substitution $u=x^3$.

3. From $r=\sqrt{x^2+y^2}$, one gets $r=\sqrt{3+1}=2$. Since $x>0$ and $y<0$ in this problem, the angle θ is in the interval $(3\pi/2, 2\pi)$. As $\tan^{-1}(y/x) = \tan^{-1}(-1/\sqrt{3}) = -\pi/6$, we get $\theta = 11\pi/6$. Therefore, $(r,\theta,z) = (2, 11\pi/6, -2)$. From $\rho=\sqrt{x^2+y^2+z^2}$, one gets $\rho = \sqrt{3+1+4} = 2\sqrt{2}$. From $z=\rho\cos\phi$, we now get $\phi = \cos^{-1}(z/\rho) = \cos^{-1}(-1/\sqrt{2}) = 3\pi/4$. Thus, $(\rho,\phi,\theta) = (2\sqrt{2}, 3\pi/4, 11\pi/6)$. Note that our current text (Spring, 2009) labels spherical coordinates as (ρ,θ,ϕ) instead of (ρ,ϕ,θ). The notation varies from book to book.

4. The region of integration is one-eighth of the circle $x^2+y^2=2$ lying between y-axis and the line $y=x$ in the first quadrant. Using $x^2+y^2=r^2$ and $dy\,dx = r\,dr\,d\theta$ leads to

   $$\int_{\pi/4}^{\pi/2}\int_0^{\sqrt{2}} r^3\cdot r\,dr\,d\theta = \int_{\pi/4}^{\pi/2}\frac{r^5}{5}\Big|_0^{\sqrt{2}} d\theta = \frac{2^{5/2}}{5}\int_{\pi/4}^{\pi/2} d\theta = \frac{2^{5/2}\pi}{20} = \frac{\sqrt{2}\,\pi}{5}.$$

5. Recalling that $z=\rho\cos\phi$ and $\sqrt{x^2+y^2} = r = \rho\sin\phi$, the equation $z = \sqrt{x^2+y^2}/\sqrt{3}$ can be written in spherical coordinates as $\rho\cos\phi = (\rho\sin\phi)/\sqrt{3}$. Rewriting this, we get $\tan\phi=\sqrt{3}$ so $\phi=\pi/3$. The equation $x^2+y^2+z^2=4$ can be written in spherical coordinates as $\rho=2$. The solid is above the cone $\phi=\pi/3$ and below the sphere $\rho=2$. The volume is

   $$\int_0^{2\pi}\int_0^{\pi/3}\int_0^2 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

6. The triple integral is over the solid above the quarter circle $x^2+y^2=1$ in the xy-plane and below the cone $z=\sqrt{x^2+y^2}$. Converting to cylindrical coordinates gives

   $$\int_0^{\pi/2}\int_0^1\int_0^r \frac{1}{r^2}\,r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1\int_0^r \frac{1}{r}\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 \frac{z}{r}\Big|_0^r dr\,d\theta$$
   $$= \int_0^{\pi/2}\int_0^1 dr\,d\theta = \int_0^{\pi/2} r\Big|_0^1 d\theta = \int_0^{\pi/2} d\theta = \frac{\pi}{2}.$$

Some 1994 Solutions:

1. (c) The region is in the first quadrant above the $y = x^2$ and below the line $y = \pi^2$. Changing the order of integration gives

$$\int_0^{\pi^2} \int_0^{\sqrt{y}} \frac{\sin\sqrt{y}}{y}\, dx\, dy = \int_0^{\pi^2} \frac{\sin\sqrt{y}}{y} \cdot x \Big|_0^{\sqrt{y}} dy = \int_0^{\pi^2} \frac{\sin\sqrt{y}}{\sqrt{y}}\, dy.$$

For this last integral, use the substitution $u = \sqrt{y}$ so that $du = dy/(2\sqrt{y})$ or $2du = (1/\sqrt{y})dy$. Since $\int 2\sin u\, du = -2\cos u + C$, we get that the above integral is equal to $-2\cos\sqrt{y}\Big|_0^{\pi^2} = -2\cos\pi + 2\cos 0 = 4$.

3. We have $r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$. Since $x > 0$ and $y > 0$ in this problem, the angle θ is in the interval $(0, \pi/2)$. As $\tan^{-1}(y/x) = \tan^{-1}(1) = \pi/4$, we get $\theta = \pi/4$. Therefore, $(r, \theta, z) = (\sqrt{2}, \pi/4, \sqrt{6})$. Here, $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1+1+6} = 2\sqrt{2}$. From $z = \rho\cos\phi$, we get $\phi = \cos^{-1}(z/\rho) = \cos^{-1}(\sqrt{3}/2) = \pi/6$. Thus, $(\rho, \phi, \theta) = (2\sqrt{2}, \pi/6, \pi/4)$.

4. The integration is over the semicircular region to the right of the y-axis and inside $x^2 + y^2 = 16$. Replacing $x^2 + y^2$ with r^2 and $dx\, dy$ with $r\, dr\, d\theta$, we get

$$\int_{-\pi/2}^{\pi/2} \int_0^4 e^{r^2}\, r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \frac{e^{r^2}}{2}\Big|_0^4 d\theta = \frac{e^{16}-1}{2} \int_{-\pi/2}^{\pi/2} d\theta = \frac{(e^{16}-1)\pi}{2}.$$

5. The equation $x^2 + y^2 + z^2 = 9$ is equivalent to the spherical equation $\rho = 3$. The equation $z = -\sqrt{x^2 + y^2}$ can be converted to spherical variables as $\rho\cos\phi = -r = -\rho\sin\phi$ which is the same as $\tan\phi = -1$ or $\phi = 3\pi/4$. Therefore, the solid is above the cone $\phi = 3\pi/4$ and inside the sphere $\rho = 3$. The volume is therefore

$$\int_0^{2\pi} \int_0^{3\pi/4} \int_0^3 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

6. The presence of numerous occurrences of $x^2 + y^2$ indicate that we should convert to cylindrical coordinates. The limits of integration for the variables x and y are over a complete circle of radius 3 centered at the origin. The triple integral is over this circle between the plane $z = 4$ and hemisphere $z = \sqrt{25 - x^2 - y^2}$. The equation $z = \sqrt{25 - x^2 - y^2}$ in cylindrical coordinates is the same as $z = \sqrt{25 - r^2}$. Therefore, the triple integral can be written as

$$\int_0^{2\pi} \int_0^3 \int_4^{\sqrt{25-r^2}} \frac{\sin r}{r(\sqrt{25-r^2}-4)}\, r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^3 \frac{\sin r}{r(\sqrt{25-r^2}-4)}\, rz \Big|_4^{\sqrt{25-r^2}} dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^3 \sin r\, dr\, d\theta = \int_0^{2\pi} (-\cos r)\Big|_0^3 d\theta = (1 - \cos 3) \int_0^{2\pi} d\theta = 2\pi(1 - \cos 3).$$

Formally, the above looks sound and is what was expected, but the integral actually requires a little bit more care as the denominator of the integrand of the triple integral is zero when $r = 3$.

Some 1998 Solutions:

1. (c) The double integral is over the triangle above $y = x$ and below $y = \pi/2$ in the first quadrant. Changing the order of integration, we get

$$\int_0^{\pi/2} \int_0^y \frac{\cos x}{\sqrt{1+\cos y}}\, dx\, dy = \int_0^{\pi/2} \left. \frac{\sin x}{\sqrt{1+\cos y}} \right|_0^y dy$$

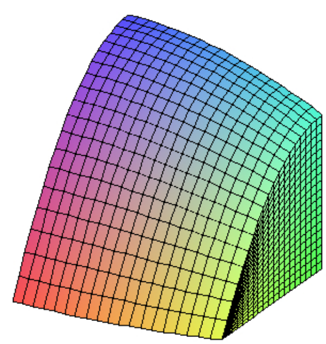
$$= \int_0^{\pi/2} \frac{\sin y}{\sqrt{1+\cos y}}\, dy = -2\sqrt{1+\cos y}\Big|_0^{\pi/2} = 2(\sqrt{2}-1).$$

The last integral can be done with a u-substitution, either $u = 1 + \cos y$ or $u = \sqrt{1+\cos y}$.

2. Here, $r = \sqrt{x^2+y^2} = \sqrt{1+0} = 1$. Since $\tan^{-1} 0 = 0$ and $x > 0$, we get $\theta = 0$ (note that if the point were $(x, y, z) = (-1, 0, 1)$, we would still have that $\tan^{-1}(y/x) = \tan^{-1} 0 = 0$ but, since $-1 < 0$, we would get $\theta = \pi$). So $(r, \theta, z) = (1, 0, 1)$. From $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{2}$ and $\phi = \cos^{-1}(z/\rho) = \cos^{-1}(1/\sqrt{2}) = \pi/4$, we get $(\rho, \theta, \phi) = (\sqrt{2}, 0, \pi/4)$.

4. The top of the solid is the paraboloid $z = 4 - x^2 - y^2$. Note that the paraboloid intersects the xy-plane (the plane $z = 0$) in the circle $x^2 + y^2 = 4$. The bottom of the solid is that part of the circle $x^2 + y^2 = 4$ (and its interior) that lies below $y = 1$ in the xy-plane. We get then that the volume can be written as

$$\int_0^1 \int_0^{\sqrt{4-y^2}} \int_0^{4-x^2-y^2} dz\, dx\, dy.$$



5. The limits on z don't need to change. The integrals for x and y are over the semicircle in the xy-plane that lies above the x-axis and below $x^2 + y^2 = 1$. Recalling that $\sqrt{x^2+y^2} = r$ and $dz\, dx\, dy = r\, dz\, dr\, d\theta$, we can rewrite the triple integral in the problem as

$$\int_0^{\pi} \int_0^1 \int_{-2}^2 r^2\, dz\, dr\, d\theta = 4 \int_0^{\pi} \int_0^1 r^2\, dr\, d\theta = 4 \int_0^{\pi} \left. \frac{r^3}{3} \right|_0^1 d\theta = \frac{4}{3} \int_0^{\pi} d\theta = \frac{4\pi}{3}.$$

6. The integration is over a hemisphere above the xy-plane of radius 2 centered at the origin. This can be seen by observing the bottom of the solid is $z = 0$ (the xy-plane), the top of the solid is $x^2 + y^2 + z^2 = 4$, and the integration in the variables x and y is over the entire circle $x^2 + y^2 \le 4$ (where the sphere intersects the xy-plane). Recall $x^2 + y^2 + z^2 = \rho^2$. Also,

$$x^2 + y^2 = r^2 = (\rho \sin\phi)^2 = \rho^2 \sin^2\phi.$$

Since $\sin\phi \ge 0$, we have $\sqrt{\sin^2\phi} = \sin\phi$ so that

$$\sqrt{(x^2+y^2)(x^2+y^2+z^2)} = \sqrt{\rho^2 \sin^2\phi \cdot \rho^2} = \sqrt{\rho^4 \sin^2\phi} = \rho^2 \sin\phi.$$

Since $dz\,dy\,dx = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$, the triple integral in the problem is the same as

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \frac{1}{\rho^2\sin\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 d\rho\, d\phi\, d\theta$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2} d\phi\, d\theta = \pi\int_0^{2\pi} d\theta = 2\pi^2.$$

This is another question where the denominator in the original integrand can be 0, so some care should be given to understand what the above is really saying. This would not be expected of you. But to clarify, in this case, one can fix $\varepsilon > 0$ and let G_ε be the solid between the hemisphere above the xy-plane of radius ε centered at the origin and the hemisphere above the xy-plane of radius 2 centered at the origin. Then the triple integral in the problem can be interpreted as

$$\lim_{\varepsilon\to 0}\iiint_{G_\varepsilon} \frac{dz\,dy\,dx}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} = \lim_{\varepsilon\to 0}\int_0^{2\pi}\int_0^{\pi/2}\int_\varepsilon^2 \frac{1}{\rho^2\sin\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \lim_{\varepsilon\to 0}\int_0^{2\pi}\int_0^{\pi/2}\int_\varepsilon^2 d\rho\, d\phi\, d\theta = \lim_{\varepsilon\to 0}(2-\varepsilon)\int_0^{2\pi}\int_0^{\pi/2} d\phi\, d\theta$$

$$= \lim_{\varepsilon\to 0}\frac{(2-\varepsilon)\pi}{2}\int_0^{2\pi} d\theta = \lim_{\varepsilon\to 0}(2-\varepsilon)\pi^2 = 2\pi^2.$$

Some 1999 Solutions:

1. (b) $\displaystyle\int_0^{\pi}\int_0^2 \theta\, dr\, d\theta = \int_0^{\pi} \theta\, r\Big|_0^2\, d\theta = \int_0^{\pi} 2\theta\, d\theta = \theta^2\Big|_0^{\pi} = \pi^2$

   (c) The double integral is over the region between $x = y^2$ and $x = 1$. In this region, $0 \le x \le 1$; also, for each x, we have $-\sqrt{x} \le y \le \sqrt{x}$. Changing the limits of integration therefore gives

$$\int_0^1\int_{-\sqrt{x}}^{\sqrt{x}} \cos(x^{3/2})\, dy\, dx = \int_0^1 \cos(x^{3/2})y\,\Big|_{-\sqrt{x}}^{\sqrt{x}}\, dx$$

$$= \int_0^1 2x^{1/2}\cos(x^{3/2})\, dx = \frac{4}{3}\sin(x^{3/2})\Big|_0^1 = \frac{4}{3}\sin(1).$$

2. (a) The double integral is over the triangle above the x-axis, below the line $y = x$ and to the left of $x = 1$. In this region, $0 \le y \le 1$. For a fixed y, we have $y \le x \le 1$ (the curve on the left of the triangle is $x = y$ and the curve on the right is $x = 1$). The answer is therefore

$$\int_0^1\int_y^1 f(x,y)\, dx\, dy.$$

   (b) The region of integration is in the first quadrant between the curves $y = x^3$ and $y = 1$. Here, $0 \le y \le 1$ and $0 \le x \le y^{1/3}$. Therefore, the integral is equivalent to

$$\int_0^1\int_0^{y^{1/3}} f(x,y)\, dx\, dy.$$

3. (a) The region of integration can be split up into three smaller regions. One region is $0 \le x \le 2$ and $0 \le y \le 2$, and the double integral over this region is the volume of a box with a 2 by 2 base and height 3 (the value of $f(x, y)$ here). Another region is $0 \le x \le 2$ and $2 < y \le 3$, and the double integral over this region is minus the volume of a 2 by 1 by 1 box (since $f(x, y) = -1$ here). The third region is $2 < x \le 3$ and $0 \le y \le 3$, and the double integral over this region is the volume of a 1 by 3 by 2 box (since $f(x, y) = 2$ here). Hence, the value ofthe double integral is $2 \cdot 2 \cdot 3 - 2 \cdot 1 \cdot 1 + 1 \cdot 3 \cdot 2 = 12 - 2 + 6 = 16$.

   (b) The value of $f(x, y)$ for $0 \le x \le 2$ and $0 \le y \le 1$ is 3. The value of $f(x, y)$ for $2 < x \le 3$ and $0 \le y \le 1$ is 2. Therefore, the double integral is the sum of the volume of a 2 by 1 by 3 box and the volume of a 1 by 1 by 2 box. The value is $2 \cdot 1 \cdot 3 + 1 \cdot 1 \cdot 2 = 8$.

4. Here, $r = \sqrt{x^2 + y^2} = \sqrt{2 + 2} = 2$ and (noting x and y are positive) $\theta = \tan^{-1}(\sqrt{2}/\sqrt{2}) = \tan^{-1}(1) = \pi/4$. Also, $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{2 + 2 + 4} = 2\sqrt{2}$ and $\phi = \cos^{-1}(z/\rho) = \cos^{-1}(1/\sqrt{2}) = \pi/4$. Therefore, $(r, \theta, z) = (2, \pi/4, 2)$ and $(\rho, \theta, \phi) = (2\sqrt{2}, \pi/4, \pi/4)$.

5. The solid is a fourth of a cylinder lying in the first octant bounded by $y^2 + z^2 = 4$ and between the planes $x = 0$ and $x = 3$. Note that the solid intersects the yz-plane in the points (y, z) lying in the quarter circle $y^2 + z^2 \le 4$ where both y and z are positive. The volume can be expressed as

$$\int_0^2 \int_0^{\sqrt{4-y^2}} \int_0^3 dx\, dz\, dy.$$

6. Any point (x_0, y_0, z_0) that is on both surfaces satisfies $z_0 = x_0^2 + y_0^2$ and $x_0^2 + y_0^2 + z_0^2 = 6$. This implies that $6 = x_0^2 + y_0^2 + z_0^2 = z_0 + z_0^2$. So $z_0^2 + z_0 - 6 = 0$, which can be rewritten as $(z_0 + 3)(z_0 - 2) = 0$. Thus, $z_0 = -3$ or $z_0 = 2$. Recalling that $z_0 = x_0^2 + y_0^2$, we see that $z_0 \ne -3$ (i.e., a negative number cannot be a sum of two nonnegative numbers). Therefore, $z_0 = 2$. This implies $2 = z_0 = x_0^2 + y_0^2$. In fact, any point (x, y) on the circle $x^2 + y^2 = 2$ is such that (x, y) satisfies both $2 = x^2 + y^2$ and $x^2 + y^2 + 2^2 = 6$. Thus, the points (x, y) on the circle $x^2 + y^2 = 2$ are precisely the points $(x, y, 2)$ that lie on both the surfaces $z = x^2 + y^2$ and $x^2 + y^2 + z^2 = 6$ in the problem. In other words, the paraboloid $z = x^2 + y^2$ opens upward and intersects the sphere $x^2 + y^2 + z^2 = 6$ above the points (x, y) in the xy-plane that are on the circle $x^2 + y^2 = 2$. The solid will lie above this circle and its interior. So this is the region we want to integrate over in the xy-plane. Converting to cylindrical coordinates, we get that the bottom surface is $z = x^2 + y^2 = r^2$ and the top surface is $z = \sqrt{6 - (x^2 + y^2)} = \sqrt{6 - r^2}$. Hence, the volume can be expressed as

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{\sqrt{6-r^2}} r\, dz\, dr\, d\theta.$$

7. Observe that $x^2 + y^2 + z^2 = 1$ and $z = 0$ (the xy-plane) intersect at the points satisfying $x^2 + y^2 = 1$ in the xy-plane. The solid is a hemisphere centered at the origin of radius 1 above $x^2 + y^2 \le 1$ in the xy-plane. Replacing $(x^2 + y^2 + z^2)^{3/2}$ in the integrand with $(\rho^2)^{3/2} = \rho^3$, and replacing dV

with $\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ allows us to rewrite the integral as

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^3 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/2} (\sin\phi) \, \frac{\rho^6}{6}\bigg|_0^1 \, d\phi \, d\theta$$

$$= \frac{1}{6} \int_0^{2\pi} \int_0^{\pi/2} \sin\phi \, d\phi \, d\theta = \frac{1}{6} \int_0^{2\pi} (-\cos\phi)\bigg|_0^{\pi/2} \, d\theta = \frac{1}{6} \int_0^{2\pi} d\theta = \frac{\pi}{3}.$$

Some 2001, Spring, Solutions:

1. (c) Observe that $y = (\pi/2) - x$ is the line with slope -1 intersecting both the x-axis and and y-axis at $\pi/2$ (that is, at $(\pi/2, 0)$ and $(0, \pi/2)$). The double integral is over the triangle below this line in the first quadrant. Interchanging the order of integration results in the equivalent integral

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}-y} x \cos\left(\left(\frac{\pi}{2} - y\right)^3\right) dx \, dy = \int_0^{\frac{\pi}{2}} \cos\left(\left(\frac{\pi}{2} - y\right)^3\right) \frac{x^2}{2}\bigg|_0^{\frac{\pi}{2}-y} dy$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos\left(\left(\frac{\pi}{2} - y\right)^3\right) \left(\frac{\pi}{2} - y\right)^2 dy = \frac{-1}{6} \sin\left(\left(\frac{\pi}{2} - y\right)^3\right)\bigg|_0^{\frac{\pi}{2}} = \frac{1}{6} \sin\left(\frac{\pi^3}{8}\right).$$

The last integral above can be done with the u-substitution $u = \big((\pi/2) - y\big)^3$ so that $du = -3\big((\pi/2) - y\big)^2 dy$.

2. Observe that x and y are negative. Since $\tan^{-1}(y/x) = \tan^{-1}(1/\sqrt{3}) = \pi/6$, we get that $\theta = \pi + (\pi/6) = 7\pi/6$. From $r = \sqrt{x^2 + y^2} = \sqrt{(3/2) + (1/2)} = \sqrt{2}$, we obtain that $(r, \theta, z) = (\sqrt{2}, 7\pi/6, -\sqrt{2})$. Since $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{(3/2) + (1/2) + 2} = 2$ and $\phi = \cos^{-1}(z/\rho) = \cos^{-1}(-\sqrt{2}/2) = 3\pi/4$, we obtain $(\rho, \theta, \phi) = (2, 7\pi/6, 3\pi/4)$.

3. Presumably, the intent here is that $R = \{(x, y) : 0 \le x \le 3, 0 \le y \le 5\}$ (the domain of $f(x, y)$). This can be divided into the three regions suggested by the definition of $f(x, y)$, namely

$$R_1 = \{(x, y) : 0 \le x \le 3, 0 \le y \le 2\}, \quad R_2 = \{(x, y) : 0 \le x \le 2, 2 < y \le 5\},$$

and

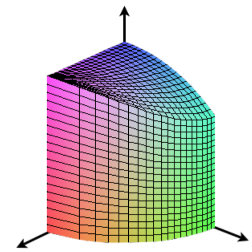
$$R_3 = \{(x, y) : 2 < x \le 3, 2 < y \le 5\}.$$

This gives

$$\iint\limits_R f(x, y) \, dA = \iint\limits_{R_1} f(x, y) \, dA + \iint\limits_{R_2} f(x, y) \, dA + \iint\limits_{R_3} f(x, y) \, dA$$

$$= -2 \cdot \int_0^3 \int_0^2 dx \, dy + 0 \cdot \int_0^2 \int_2^5 dx \, dy + 5 \cdot \int_2^3 \int_2^5 dx \, dy = -2 \cdot 6 + 0 \cdot 6 + 5 \cdot 3 = 3.$$

5. (a) The bottom of the solid is the quarter of the circle $x^2 + y^2 \leq 1$ that lies in the first quadrant of the xy-plane. The top of the solid is the cylinder $y^2 + z^2 = 2$. Hence, the volume can be written as



$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{2-y^2}} dz\, dy\, dx.$$

(b) For the quarter of the circle described in part (a), we have $0 \leq \theta \leq \pi/2$ and $0 \leq r \leq 1$. Since $y = r \sin\theta$ in cylindrical coordinates, we obtain $2 - y^2 = 2 - r^2 \sin^2\theta$. Thus, the triple integral in cylindrical coordinates is

$$\int_0^{\pi/2} \int_0^1 \int_0^{\sqrt{2-r^2\sin^2\theta}} r\, dz\, dr\, d\theta.$$

6. Note that the inner integral goes from $x = -\sqrt{16 - y^2 - z^2}$ to $x = \sqrt{16 - y^2 - z^2}$. In other words, it goes from the back side of the sphere $x^2 + y^2 + z^2 = 16$ to the front side of it. The remaining integrals are over the quarter circle in the yz-plane given by $y^2 + z^2 \leq 16$ with $y \geq 0$ and $z \geq 0$. So the solid is the quarter of a sphere centered at the origin of radius 4 that lies above the xy-plane and to the right of the xz-plane. Using $x^2+y^2+z^2 = \rho^2$ and $dx\, dy\, dz = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$, we can rewrite the given triple integral as

$$\begin{aligned}
\int_0^{\pi} \int_0^{\pi/2} \int_0^4 (\rho^2)^{7/2} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta &= \int_0^{\pi} \int_0^{\pi/2} \int_0^4 \rho^9 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= \int_0^{\pi} \int_0^{\pi/2} (\sin\phi) \left.\frac{\rho^{10}}{10}\right|_0^4 d\phi\, d\theta = \frac{4^{10}}{10} \int_0^{\pi} \int_0^{\pi/2} \sin\phi\, d\phi\, d\theta \\
&= \frac{4^{10}}{10} \int_0^{\pi} (-\cos\phi) \Big|_0^{\pi/2} d\theta = \frac{4^{10}}{10} \int_0^{\pi} d\theta = \frac{4^{10}\pi}{10} = \frac{2^{19}\pi}{5}.
\end{aligned}$$