Old Math 241 Test 2's

Some 1992 Solutions:

3. The chain rule gives

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} \cdot \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial t} = y^2\sqrt{y}\cos(x+y) \cdot 3 \cdot 2 \cdot 3w = 18wy^2\sqrt{y}\cos(x+y).$$

4. We want both $\partial z/\partial x$ and $\partial z/\partial y$ to be 0 at P. Calculating the partial derivatives, we obtain

$$3(x+y)^2x + (x+y)^3 + 2x - 1 = 0 \quad \text{and} \quad 3(x+y)^2x = 0.$$

The second equation gives that either $x = 0$ or $y = -x$. If $x = 0$, then the first equation gives $y^3 - 1 = 0$ so that $y = 1$. If $y = -x$, then the first equation gives $2x - 1 = 0$ so that $x = 1/2$ and $y = -x = -1/2$. To get the z-coordinate of each point P, we plug in our values of x and y into the equation $z = (x+y)^3x + x^2 - x$. We deduce that there are two such points P, namely $(0, 1, 0)$ and $(1/2, -1/2, -1/4)$.

5. Here, we have

$$f_x = 2x + 2y + 2, \quad f_y = 2x + 4y, \quad f_{xx} = 2, \quad f_{yy} = 4 \quad \text{and} \quad f_{xy} = 2.$$

We want points where both $f_x = 0$ and $f_y = 0$. Since $f_y - f_x = 2y - 2$, we deduce $y = 1$. Taking $y = 1$ in the equation $2x + 4y = 0$, we get $x = -2$. As $z = f(x, y)$, for $x = -2$ and $y = 1$, we have $z = -1$. So there is one point, namely $(-2, 1, -1)$, to consider for this problem. We have $D = 2 \cdot 4 - 2^2 = 4$ and $f_{xx} > 0$, so there is a relative minimum at $(-2, 1, -1)$.

6. Since $f_x = y^2 + 5$ is never 0, there are no points inside the disk to consider, and we need only consider the boundary. Using $y^2 = 4 - x^2$ on the boundary, we deduce that if (x, y) is on the boundary then $f(x, y) = w(x)$ where

$$w(x) = x(4 - x^2) + 3(4 - x^2) + 5x - 5 = -x^3 - 3x^2 + 9x + 7.$$

Since $x^2 + y^2 = 4$, we are interested in finding the extrema for $w(x)$ with $-2 \le x \le 2$. As $w'(x) = -3x^2 - 6x + 9 = -3(x-1)(x+3)$ and -3 is not in the interval $[-2, 2]$, we are interested in $w(x)$ for $x \in \{-2, 1, 2\}$ (note that the endpoints of the interval $[-2, 2]$ are included here). Since

$$w(-2) = 8 - 12 - 18 + 7 = -15, \; w(1) = -1 - 3 + 9 + 7 = 12, \; \text{and } w(2) = -8 - 12 + 18 + 7 = 5,$$

we deduce that the (global) maximum value is 12 and the (global) minimum value is -15.

Some 1994 Solutions:

1. Since $f(0, y) = 0$, the function approaches (indeed equals) 0 as (x, y) approaches the origin along the line $x = 0$ (the y-axis). Since $f(x, x) = 1/2$, the function approaches (indeed equals) $1/2$ as (x, y) approaches the origin along the line $y = x$. Since $0 \ne 1/2$, these two limiting values are not equal, and we deduce that $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

2. Using $\|\overrightarrow{v}\| = \sqrt{10}$, we obtain that $\overrightarrow{u} = \langle 1/\sqrt{10}, 3/\sqrt{10}\rangle$ is a unit vector in the direction of $\overrightarrow{v}$. The problem is to determine the value of $D_{\overrightarrow{u}} f(1,-1)$. Now, using $\nabla f = \langle 2xy, x^2+2y\rangle$, we obtain that $\nabla f(1,-1) = \langle -2,-1\rangle$ and, hence, $D_{\overrightarrow{u}} f(1,-1) = \nabla f(1,-1)\bullet\overrightarrow{u} = -5/\sqrt{10}$.

3. We take $F(x,y,z) = x^2 - 2y^2 - xyz^2$. Since $\nabla F = \langle 2x - yz^2, -4y - xz^2, -2xyz\rangle$, we get $\nabla F(1,-1,-1) = \langle 3,3,-2\rangle$. Therefore, the tangent plane is of the form $3x+3y-2z = k$. Since $(1,-1,-1)$ is on the plane, we deduce $k = 3-3+2 = 2$ and the plane is $3x+3y-2z = 2$.

5. The chain rule gives

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = (3x^2y + y^2 - 1)\cdot(e^s t) + (x^3 + 2xy)\cdot(t^2 e^{st})$$
$$= \left(3t^3e^{2s+st} + t^2e^{2st} - 1\right)\cdot(e^s t) + \left(t^3e^{3s} + 2t^2e^{s+st}\right)\cdot(t^2e^{st}).$$

6. Setting $f_x = 12x - 6 = 0$ and $f_y = 6y = 0$, we deduce that $(x,y) = (1/2,0)$. Since $(1/2,0)$ is in R, we consider $f(1/2,0) = 3/2 - 3 - 9 = -21/2$. On the boundary of R, we have $y^2 = 4 - x^2$ so that $f(x,y) = 6x^2 + 3(4-x^2) - 6x - 9 = 3x^2 - 6x + 3$. Given $-2 \le x \le 2$ on the boundary, we are interested in the extrema of $w(x) = 3x^2 - 6x + 3$ on the interval $[-2,2]$. Since $w'(x) = 6x - 6$, we consider $w(x)$ at $x = 1$ and at the endpoints of our interval $[-2,2]$. We have $w(-2) = 27$, $w(1) = 0$ and $w(2) = 3$. These are possible extrema for $f(x,y)$ on the boundary of R. Recalling $f(1/2,0) = -21/2$, we deduce that the global maximum value of $f(x,y)$ on R is 27 and the global minimum value of $f(x,y)$ on R is $-21/2$.

7. We have

$$f_x = (3y^4+1)(2x-2), \quad f_y = 12y^3(x^2-2x+2) - 36y^2 + 24y,$$
$$f_{xx} = 2(3y^4+1), \quad f_{yy} = 36y^2(x^2-2x+2) - 72y + 24, \quad \text{and} \quad f_{xy} = 12y^3(2x-2).$$

Setting $f_x = 0$, we deduce $x = 1$. Taking $x = 1$ in the equation $f_y = 0$ gives $12y^3 - 36y^2 + 24y = 0$. Since $12y^3 - 36y^2 + 24y = 12y(y-1)(y-2)$, we deduce that the critical points are $(x,y) = (1,0)$, $(1,1)$ and $(1,2)$. Since $D = f_{xx}f_{yy} - f_{xy}^2$, we get

$$D(1,0) = 2\cdot 24 - 0^2 > 0, \quad D(1,1) = 8\cdot(-12) - 0^2 < 0, \quad \text{and} \quad D(1,2) = 98\cdot 24 - 0^2 > 0.$$

Noting the signs of f_{xx} at these points (all positive), there is a local minimum at $(1,0)$ and at $(1,2)$ and a saddle point at $(1,1)$.

Some 1998 Solutions:

1. (a) Since $f_x = 2x$ and $f_y = -2y$, we get $\nabla f(0,1) = \langle 0,-2\rangle$. A unit vector in the direction of $\langle 1,1\rangle$ is $\overrightarrow{u} = \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$. Thus, the answer is $D_{\overrightarrow{u}} f(0,1) = \langle 0,-2\rangle\bullet\langle 1/\sqrt{2}, 1/\sqrt{2}\rangle = -\sqrt{2}$.

   (b) The maximum value of the directional derivative at $(0,1)$ is $\|\nabla f(0,1)\| = \|\langle 0,-2\rangle\| = 2$.

3. Taking $F(x,y,z) = x^3 - x\sin(y) + z^2$, we get

$$\nabla F = \langle 3x^2 - \sin y, -x\cos y, 2z\rangle \quad \text{and} \quad \nabla F(-1,0,1) = \langle 3,1,2\rangle.$$

Hence, we can use $3x + y + 2z = k$ for the equation of the plane where $k = -3+2 = -1$ (obtained from the fact that $(-1,0,1)$ is on the plane). Thus, the plane is $3x + y + 2z = -1$.

6. (a) On the circle, $x^2 + y^2 = 4$ so that

$$f(x, y) = 9x^2 + 6y^2 + 6x + 4 = 3x^2 + 6(x^2 + y^2) + 6x + 4 = 3x^2 + 24 + 6x + 4 = 3x^2 + 6x + 28.$$

Since $-2 \le x \le 2$ on the circle, we are interested in finding the extrema of $w(x) = 3x^2 + 6x + 28$ where $-2 \le x \le 2$. Since $w'(x) = 6x + 6 = 0$ precisely when $x = -1$ and since -1 is in the interval $[-2, 2]$, we are left with comparing the numbers $w(-2) = 28$, $w(-1) = 25$ and $w(2) = 52$. Therefore, on the circle, the global maximum value is 52 and the global minimum value is 25.

(b) Observe that $f_x = 18x + 6 = 6(3x + 1) = 0$ and $f_y = 12y = 0$ precisely when $x = -1/3$ and $y = 0$. Also, the point $(-1/3, 0)$ is in R. So we consider $f(-1/3, 0) = 1-2+4 = 3$ and compare this with the extrema we already found on the boundary of R in part (a). The global maximum on R is 52, and the global minimum is 3.

7. We have

$$f_x = 4xy - 8y, \quad f_y = 2x^2 - 8x + 2y, \quad f_{xx} = 4y, \quad f_{yy} = 2, \quad \text{and} \quad f_{xy} = 4x - 8.$$

Setting $f_x = 0$, we see that $4y(x - 2) = 0$ so that either $x = 2$ or $y = 0$. Putting $x = 2$ in $f_y = 0$, we deduce $-8 + 2y = 0$ and, hence, $y = 4$. Putting $y = 0$ in $f_y = 0$, we deduce $2x^2 - 8x = 2x(x - 4) = 0$ and, hence, $x = 0$ or $x = 4$. This gives us three points to consider, namely $(2, 4)$, $(0, 0)$ and $(4, 0)$. Since $D = f_{xx}f_{yy} - f_{xy}^2$, we get

$$D(2, 4) = 16 \cdot 2 - 0^2 > 0, \quad D(0, 0) = 0 \cdot 2 - (-8)^2 < 0, \quad \text{and} \quad D(4, 0) = 0 \cdot 2 - 8^2 < 0.$$

Since $f_{xx}(2, 4) = 16 > 0$, there is a local minimum at $(2, 4)$; there are saddle points at $(0, 0)$ and $(4, 0)$.

Some 1999 Solutions:

6. Since $f_x = 3+y^2$ cannot equal 0, the critical points are simply the points on the boundary of S, that is the points (x, y) satisfying $x^2+y^2 = 9$. On the boundary, $f(x, y) = 3x+x(9-x^2) = -x^3+12x$. We set $w(x) = -x^3 + 12x$ where x is in the interval $[-3, 3]$ (this is the interval x lies on for (x, y) in S). Since $w'(x) = -3x^2 + 12 = -3(x^2 - 4) = 0$ precisely when $x = \pm 2$ and -2 and 2 are in $[-3, 3]$, we need only consider $w(-3) = 27 - 36 = -9$, $w(-2) = 8 - 24 = -16$, $w(2) = -8 + 24 = 16$ and $w(3) = -27 + 36 = 9$. Note that when $x = \pm 2$ on the boundary $x^2 + y^2 = 9$, we have $y = \pm\sqrt{5}$. Therefore, the (global) maximum value of $f(x, y)$ on S is 16 and it occurs at $(x, y) = (2, \pm\sqrt{5})$, and the (global) minimum value of $f(x, y)$ on S is -16 and it occurs at $(x, y) = (-2, \pm\sqrt{5})$.

7. We have

$$f_x = 4x^3 + 4y + y^2, \quad f_y = 4x + 2xy, \quad f_{xx} = 12x^2, \quad f_{yy} = 2x, \quad \text{and} \quad f_{xy} = 4 + 2y.$$

Since $4x + 2xy = 2x(2 + y)$, we want to consider points where $x = 0$ or $y = -2$. Setting $x = 0$ in $f_x = 0$ gives $4y + y^2 = y(4 + y) = 0$ so that $y = 0$ or $y = -4$. Setting $y = -2$ in $f_x = 0$ gives

$4x^3 - 4 = 4(x^3 - 1) = 0$ so that $x = 1$. Hence, the three critical points are $(0,0)$, $(0,-4)$ and $(1,-2)$. Using $D = f_{xx}f_{yy} - f_{xy}^2$, we obtain

$$D(0,0) = 0 \cdot 0 - 4^2 < 0, \quad D(0,-4) = 0 \cdot 0 - (-4)^2 < 0, \quad \text{and} \quad D(1,-2) = 12 \cdot 2 - 0^2 > 0.$$

Thus, there are saddle points at $(0,0)$ and $(0,-4)$ and a relative minimum at $(1,-2)$.

Some Spring 2001 Solutions:

1. (a) Since $\|\langle -3, 4\rangle\| = 5$, a unit vector going in the direction of $\langle -3, 4\rangle$ is $\overrightarrow{u} = \langle -3/5, 4/5\rangle$. Using $\nabla f = \langle 3x^2y^2, 2x^3y - 1\rangle$, we obtain $\nabla f(1,-2) = \langle 12, -5\rangle$. We deduce that the answer is $D_{\overrightarrow{u}} f(1,-2) = \langle 12, -5\rangle \bullet \langle -3/5, 4/5\rangle = (-36 - 20)/5 = -56/5$.

   (b) Since $\|\nabla f(1,-2)\| = \|\langle 12, -5\rangle\| = \sqrt{144 + 25} = \sqrt{169} = 13$, the minimal value of the directional derivative at $(1,-2)$ is -13.

3. The chain rule gives

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} = 3x^2y \cdot 4u^3 + (x^3 - 2y) \cdot \big(- v\sin(uv)\big).$$

6. The value of $f_x = 3x^2 + 4y^2$ is a sum of two squares (that is, $(\sqrt{3}x)^2$ and $(2y)^2$) and can only be 0 if each of the squares is 0. So $f_x = 0$ implies that $(x,y) = (0,0)$. Since $f_y = 8xy = 0$ at $(0,0)$ as well, we get that $(0,0)$ is a critical point. The other critical points are all the points on the boundary of S, that is the points (x,y) where $x^2 + y^2 = 9$. To determine the maximum and minimum values of f on S, we consider the value of f at the critical points. First, $f(0,0) = 0$. Next, we look at $f(x,y)$ where $x^2 + y^2 = 9$. Here, $f(x,y) = w(x)$ where $w(x) = x^3 + 4x(9 - x^2) = -3x^3 + 36x$ and $-3 \le x \le 3$. Since $w'(x) = -9x^2 + 36 = -9(x+2)(x-2)$ and both -2 and 2 are in the interval $[-3,3]$, we consider $w(-3) = -27$, $w(-2) = 24 - 72 = -48$, $w(2) = -24 + 72 = 48$ and $w(3) = 27$. We deduce that the maximum value of $f(x,y)$ on S is 48 and the minimum value is -48. These occur when $x = \pm 2$ on the boundary. In each case ($x = 2$ and $x = -2$), the value of y on the boundary is $y = \pm\sqrt{5}$ (since we have $x^2 + y^2 = 9$ on the boundary). This gives that the maximum occurs at $(2, \pm\sqrt{5})$ and the minimum occurs at $(-2, \pm\sqrt{5})$.

7. Observe that

$$f_x = y^3 - 6x^2 + 4x - 3y, \quad f_y = 3xy^2 - 3x,$$
$$f_{xx} = -12x + 4, \quad f_{yy} = 6xy, \quad \text{and} \quad f_{xy} = 3y^2 - 3.$$

   Since

$$f_x(1,-1) = 0, \quad f_y(1,-1) = 0, \quad f_x(-1,1) = -12, \quad f_y(-1,1) = 0,$$
$$f_x(0,0) = 0, \quad \text{and} \quad f_y(0,0) = 0,$$

   we see that $(1,-1)$ and $(0,0)$ are critical points and that $(-1,1)$ is not. Using $D = f_{xx}f_{yy} - f_{xy}^2$, we get $D(1,-1) = (-8)(-6) - 0^2 > 0$ and $D(0,0) = 4 \cdot 0 - (-3)^2 < 0$. Since $-8 < 0$, we deduce that there is a relative maximum at $(1,-1)$. There is a saddle point at $(0,0)$.