Old Math 241 Test 1's

Some 1992 Solutions:

3. The point $(1,2,2)$ is on the first given line and, hence, on $\mathcal{P}$. The vector $\overrightarrow{u} = \langle 1,0,1\rangle$ is parallel to this line and, hence, $\mathcal{P}$. The vector $\overrightarrow{v} = \langle 0,1,1\rangle$ is perpendicular to $y+z=4$ and, hence, parallel to $\mathcal{P}$. So
$$\overrightarrow{u} \times \overrightarrow{v} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \langle -1,-1,1\rangle$$
is a normal to $\mathcal{P}$. An equation for $\mathcal{P}$ is $-x-y+z=-1$ or, equivalently, $x+y-z=1$.

4. The two vectors $\overrightarrow{u} = \langle 1,-1,-1\rangle$ and $\overrightarrow{v} = \langle 1,1,0\rangle$ (perpendicular to the planes) are perpendicular to the line of intersection of the planes. A vector going in the direction of this intersection is
$$\overrightarrow{u} \times \overrightarrow{v} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & -1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \langle 1,-1,2\rangle.$$

Some 1994 Solutions:

4. From the cosine formula for dot products, we have
$$\cos\theta = \frac{\overrightarrow{u} \bullet \overrightarrow{v}}{\|\overrightarrow{u}\|\,\|\overrightarrow{v}\|} = \frac{3}{3\cdot\sqrt{2}} = \frac{1}{\sqrt{2}},$$
so $\theta = \pi/4$.

5. Using the formula for a projection,
$$\text{proj}_{\overrightarrow{v}}\,\overrightarrow{u} = \frac{\overrightarrow{u} \bullet \overrightarrow{v}}{\|\overrightarrow{v}\|^2}\,\overrightarrow{v} = \frac{3}{\sqrt{2}^2}\langle 1,0,1\rangle = \langle 3/2, 0, 3/2\rangle.$$

9. The area of the parallelogram is $\|\overrightarrow{PQ} \times \overrightarrow{PR}\|$. As $\overrightarrow{PQ} = \langle -5,1,0\rangle$ and $\overrightarrow{PR} = \langle -5,0,1\rangle$,
$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -5 & 1 & 0 \\ -5 & 0 & 1 \end{vmatrix} = \langle 1,5,5\rangle.$$
Hence, the area is $\sqrt{1+25+25} = \sqrt{51}$.

10. First, find two points that lie on both $\mathcal{P}_1$ and $\mathcal{P}_2$. Taking $x = 2$, for example, we see that we can take $z = 1$ and $y = 0$. So the point $A = (2, 0, 1)$ is on the line of intersection of $\mathcal{P}_1$ and $\mathcal{P}_2$. Similarly, taking $x = 4$, we get that the point $B = (4, 2, 0)$ is on the line. The plane we are interested in contains these two points and the given point $C = (1, 1, 1)$. Thus, we need only calculate an equation for the plane containing A, B and C. Since $\overrightarrow{AB} = \langle 2, 2, -1 \rangle$ and $\overrightarrow{AC} = \langle 1, -1, 0 \rangle$, we have

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 2 & 2 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \langle -1, -1, -4 \rangle.$$

We can use $\langle 1, 1, 4 \rangle$ then for our normal vector to the plane to get the equation $x + y + 4z = 6$.

Some 1998 Solutions:

2. (a) As $\overrightarrow{QP} = \langle 1, 4, 8 \rangle$ and $\overrightarrow{QR} = \langle 0, -6, -8 \rangle$, the area is $\frac{1}{2} \|\langle 1, 4, 8 \rangle \times \langle 0, -6, -8 \rangle\|$. Since

$$\langle 1, 4, 8 \rangle \times \langle 0, -6, -8 \rangle = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 4 & 8 \\ 0 & -6 & -8 \end{vmatrix} = \langle 16, 8, -6 \rangle,$$

the area is $\sqrt{256 + 64 + 36}/2 = \sqrt{356}/2 = \sqrt{89}$.

(b) Since

$$\cos(\angle PQR) = \frac{\overrightarrow{QP} \bullet \overrightarrow{QR}}{\|\overrightarrow{QP}\| \, \|\overrightarrow{QR}\|} = \frac{-88}{9 \cdot 10} = -\frac{44}{45},$$

the answer is $\cos^{-1}(-44/45)$.

4. (a) The vector $\overrightarrow{v} = \langle 2, 1, -1 \rangle$ is perpendicular to the plane and, therefore, parallel to the line. Thus, the answer is

$$\begin{aligned} x &= -1 + 2t \\ y &= 2 + t \\ z &= -t \end{aligned}$$

(b) The line can go in any direction that is perpendicular to the vector $\overrightarrow{v} = \langle 2, 1, -1 \rangle$ from part (a). Since $\overrightarrow{u} = \langle 0, 1, 1 \rangle$ is such that $\overrightarrow{u} \bullet \overrightarrow{v} = 0$, we can use $\overrightarrow{u}$ for the direction of the line. Therefore, one of many answers is

$$\begin{aligned} x &= -1 \\ y &= 2 + t \\ z &= t \end{aligned}$$

Some 1999 Solutions:

1. (e) One answer can be obtained by choosing S so that $\overrightarrow{PQ} = \overrightarrow{RS}$. Such an S has the property that the distances PQ and RS are equal and the lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ are parallel. Taking $S = (x, y, z)$, the equation $\overrightarrow{PQ} = \overrightarrow{RS}$ gives $\langle -2, -3, -1\rangle = \langle x - 7, y - 2, z + 3\rangle$. So $S = (5, -1, -4)$ is one answer.

3. Let $\mathcal{Q}$ be the plane we want that is perpendicular to $\mathcal{P}$ and passes through $A = (1, 4, -3)$ and $B = (1, 5, -2)$. The vector $\overrightarrow{u} = \langle 1, 1, -1\rangle$ is perpendicular to $\mathcal{P}$ and, therefore, parallel to $\mathcal{Q}$. The vector $\overrightarrow{AB} = \langle 0, 1, 1\rangle$ is also parallel to $\mathcal{Q}$. Therefore, the vector

$$\overrightarrow{u} \times \overrightarrow{AB} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \langle 2, -1, 1\rangle.$$

is normal to $\mathcal{Q}$. Thus, an equation for $\mathcal{Q}$ is $2x - y + z = -5$.

4. (a) See the answers to this test for a detailed explanation for this part.

   (b) The vector $\overrightarrow{u} = \langle 1, 0, 1\rangle$ is parallel to ℓ_1, and the vector $\overrightarrow{v} = \langle 0, 2, 1\rangle$ is parallel to ℓ_2. The vector

$$\overrightarrow{u} \times \overrightarrow{v} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \langle -2, -1, 2\rangle$$

is perpendicular to both $\overrightarrow{u}$ and $\overrightarrow{v}$. The point $P = (2, 0, -1)$ is on ℓ_1 and the point $Q = (3, 0, 1)$ is on ℓ_2. The distance between the lines is obtained by computing

$$\|\text{proj}_{\langle -2,-1,2\rangle}\overrightarrow{PQ}\| = \frac{|\langle -2, -1, 2\rangle \bullet \langle 1, 0, 2\rangle|}{\sqrt{4 + 1 + 4}} = \frac{2}{3}.$$

Some 2001 Spring Solutions:

6. (a) Since $\overrightarrow{n} = \langle 1, -1, 2\rangle$ is perpendicular to the plane $\mathcal{P}$, this vector is parallel to ℓ. Therefore, parametric equations for the line are

$$\begin{aligned} x &= t \\ y &= -t \\ z &= 1 + 2t \end{aligned}$$

   (b) Using the parametric equations for the line in the equation for the plane to find the point of intersection gives

$$t - (-t) + 2(1 + 2t) = 3 \quad \text{so that} \quad 6t = 1 \quad \text{and} \quad t = 1/6.$$

Putting this value of t into the parametric equations for the line gives the intersection point of the line and the plane, namely $(1/6, -1/6, 4/3)$.

7. (a) Replacing t with s in ℓ_2 and equating gives

$$1 + t = s, \quad 2 - t = 5 + s, \quad t = -1 + s.$$

The first two of these imply $s - t = 1$ and $s + t = -3$. Adding these two equations leads to $s = -1$. Plugging this into any of the other equations here gives $t = -2$. The value of $t = -2$ in ℓ_1 and the value of $s = -1$ in ℓ_2 both give the point $(-1, 4, -2)$, so the lines intersect at $(-1, 4, -2)$ (and no where else).

(b) Vectors going in the direction of ℓ_1 and ℓ_2, respectively, are $\overrightarrow{v}_1 = \langle 1, -1, 1 \rangle$ and $\overrightarrow{v}_2 = \langle 1, 1, 1 \rangle$. If θ is the angle between these two vectors, then

$$\cos\theta = \frac{\overrightarrow{v}_1 \bullet \overrightarrow{v}_2}{\|\overrightarrow{v}_1\| \, \|\overrightarrow{v}_2\|} = \frac{1}{3}.$$

Therefore, the smallest angle between the lines is $\cos^{-1}(1/3)$. Note that the smallest angle between two lines is necessarily in the interval $[0, \pi/2]$. Since $1/3 > 0$, the answer $\cos^{-1}(1/3)$ is in this interval.