Math 241: Test 1 Spring, 2020 SOLUTIONS

(1) Let $A = (2, 4, -1)$, $B = (-2, 8, 1)$ and $C = (6, -1, 2)$. Fill in the boxes below by doing an appropriate calculation. As noted above, your work will be graded.

(a) The distance from A to the y-axis is $\boxed{\sqrt{5}}$. Simplify.

Solution: The distance from the y-axis is determined by the x and z coordinates of the point. The answer comes from the computation $\sqrt{2^2 + (-1)^2} = \sqrt{5}$. ■

(b) $\overrightarrow{AB} = \boxed{\langle -4, 4, 2\rangle}$ and $\overrightarrow{AC} = \boxed{\langle 4, -5, 3\rangle}$. Simplify.

Solution: Here, the relevant computations are

$$\overrightarrow{AB} = \langle (-2) - 2, 8 - 4, 1 - (-1)\rangle = \langle -4, 4, 2\rangle$$

and

$$\overrightarrow{AC} = \langle 6 - 2, (-1) - 4, 2 - (-1)\rangle = \langle 4, -5, 3\rangle. \blacksquare$$

(c) The distance from A to B is $\boxed{6}$. Simplify.

Solution: This is just

$$\| \overrightarrow{AB} \| = \sqrt{(-4)^2 + 4^2 + 2^2} = \sqrt{36} = 6. \blacksquare$$

(d) A vector which has the opposite direction as $\overrightarrow{AB}$ and which has length 3 is $\boxed{\langle 2, -2, -1\rangle}$. Simplify so the answer only involves integers (no fractions).

Solution: The vector $\overrightarrow{AB}$ has length 6, so $(1/2)\,\overrightarrow{AB}$ has length 3. We want a vector of length 3 going in the opposite direction as $\overrightarrow{AB}$, so we multiply by -1 to get $-(1/2)\,\overrightarrow{AB} = -(1/2)\,\langle -4, 4, 2\rangle = \langle 2, -2, -1\rangle$. ■

(e) The measure of $\angle BAC$ is $\boxed{3\pi/4}$. Simplify. Do <u>not</u> use an inverse trig function.

Solution: We want the angle θ between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. We calculate

$$\cos\theta = \frac{\overrightarrow{AB} \bullet \overrightarrow{AC}}{\| \overrightarrow{AB} \| \, \| \overrightarrow{AC} \|} = \frac{-30}{6\sqrt{50}} = \frac{-5}{\sqrt{50}} = \frac{-5}{5\sqrt{2}} = \frac{-1}{\sqrt{2}}.$$

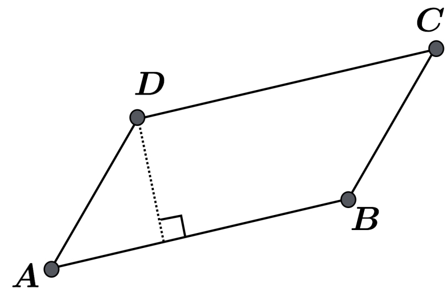
Therefore, $\theta = 3\pi/4$. ■

(2) The four vertices of a parallelogram are given by

$$A = (0, 0, 0), \quad B = (2, 1, 2),$$

$$C = (4, 1, 6) \quad \text{and} \quad D = (2, 0, 4).$$

The sides are $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{AD}$.



(not drawn to scale)

(a) Using the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$, compute the area of the parallelogram.

Area: $\boxed{6}$ (Simplify)

Solution: The area of the parallelogram is $\| \overrightarrow{AB} \times \overrightarrow{AD} \|$. So we compute

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 2 \\ 2 & 0 & 4 \end{vmatrix} = \langle 4, -4, -2 \rangle.$$

The answer is $\| \overrightarrow{AB} \times \overrightarrow{AD} \| = \sqrt{36} = 6$. ■

(b) What is the height of the parallelogram using $\overline{AB}$ as the base?

Height: $\boxed{2}$ (Simplify)

Solution: Let h be the height we want. Since the area of a parallelogram is its height multiplied by the length of its base, we see form part (a) that

$$6 = h \| \overrightarrow{AB} \| = h\sqrt{2^2 + 1^2 + 2^2} = h\sqrt{9} = 3h,$$

so $h = 2$. ■

(3) The plane $\mathcal{P}_1$ is given by the equation

$$6x - 3y + 2z = 2,$$

and the plane $\mathcal{P}_2$ is given by the equation

$$6x - 3y + 2z = 6.$$

The planes $\mathcal{P}_1$ and $\mathcal{P}_2$ are parallel (you do not need to justify this). Using the length of a projection vector, calculate the distance between the two planes. For full credit, your calculation for the answer must involve calculating the length of a projection.

Distance between planes: $\boxed{4/7}$ (Simplify)

Solution: We pick a point A on $\mathcal{P}_1$ and a point B on $\mathcal{P}_2$. There are many choices, and any will do. We take $A = (0, 0, 1)$ and $B = (1, 0, 0)$. Then $\overrightarrow{AB} = \langle 1, 0, -1 \rangle$. Each plane has a normal vector $\overrightarrow{n} = \langle 6, -3, 2 \rangle$. The distance between the planes is given by

$$\| \operatorname{proj}_{\overrightarrow{n}} \overrightarrow{AB} \| = \frac{|\overrightarrow{n} \bullet \overrightarrow{AB}|}{\| \overrightarrow{n} \|} = \frac{4}{7}. \blacksquare$$

(4) (a) Write parametric equations for the line ℓ that passes through the point $P = (1, -1, 1)$ and is parallel to the line with parametric equations $x = 2 - t$, $y = 3t$ and $z = 1 + t$.

Parametric Equations of Line ℓ: $\boxed{\begin{aligned} x &= 1 - t \\ y &= -1 + 3t \\ z &= 1 + t \end{aligned}}$

Solution: Since $P = (1, -1, 1)$ is on ℓ and the vector $\vec{v} = \langle -1, 3, 1 \rangle$ is parallel to ℓ, the answer follows. ■

(b) At what point does your answer to part (a) intersect the plane $3x + y + z = 0$?

Point of Intersection: $\boxed{(4, -10, -2)}$ (Simplify)

Solution: We want to find the value of t for which the point $(1 - t, -1 + 3t, 1 + t)$ on ℓ is also on the plane $3x + y + z = 0$. So we want

$$3(1 - t) + (-1 + 3t) + (1 + t) = 0.$$

This simplifies to $3 + t = 0$, so $t = -3$. The answer $(4, -10, -2)$ follows by plugging $t = -3$ into $(1 - t, -1 + 3t, 1 + t)$. ■

(5) The graphs for the equations below are similar to the graphs given in the "Graph Section" below. The orientation and scaling may be different. For each equation, indicate which graph in the Graph Section best matches it. For example, if the equation is for a hyperbolic paraboloid, then the graph you choose should be a hyperbolic paraboloid. Indicate your choice by writing down and circling the corresponding letter from the Graph Section next to the problem - next to (i), (ii) or (iii) below. Next, read the question in the Graph Section corresponding to the graph you choose. Then answer the question for the graph of the equation in the problem below. Do **NOT** answer the question for the graph in the Graph Section (since it may be oriented differently than the graph of the equation below).

(i) $3x^2 - y^2 - 4z^2 + 4 = 0$

Solution: The equation is equivalent to $(-3/4)x^2 + (1/4)y^2 + z^2 = 1$ (with one minus sign on the left), so the graph is $\boxed{\text{(d)}}$, a hyperboloid of one sheet. When $z = 1$, we get $3x^2 - y^2 = 0$ so that $y = \pm\sqrt{3}\,x$. Wet get $\boxed{\text{two lines}}$. ■

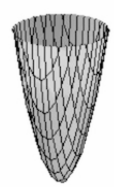
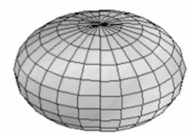
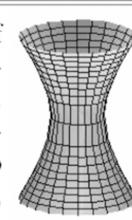
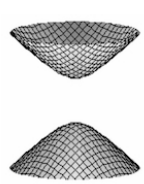
(ii) $3x^2 - y^2 - 4z^2 = 0$

Solution: This is $\boxed{\text{(c)}}$, an elliptic cone. The xy-plane is the plane $z = 0$. When $z = 0$, we get $3x^2 - y^2 = 0$ which again gives $y = \pm\sqrt{3}\,x$. Thus, we can write our parametric equations as $\boxed{x = t,\ y = \pm\sqrt{3}\,t \text{ and } z = 0}$. ■

(iii) $3x^2 - y^2 - 4z + 4 = 0$

Solution: Since a variable appears to only the first power, this is a paraboloid. As the squared terms are on the same side of the equation and have opposite signs, this is $\boxed{\text{(f)}}$, a hyperbolic paraboloid. When $y = 2020$, we have $3x^2 - 2020^2 - 4z + 4 = 0$ or equivalently $4z = 3x^2 - 2020^2 + 4 = 0$, which is $\boxed{\text{a parabola}}$. ■

Graph Section

(a) To the right is a graph of an elliptic paraboloid. What point is the vertex in the graph for the problem? Give all 3 of the coordinates. 	(b) This is a graph of an ellipsoid. What are the 2 points where the graph for the problem intersects the y-axis? Give all 3 coordinates for each point. 
(c) This is an elliptic cone. The xy-plane intersects the graph for the problem in two lines. Write down parametric equations for these two lines. 	(d) This is a hyperboloid of 1 sheet. What is the intersection of the plane $z = 1$ with the graph in the problem? Answer a point, two lines, an ellipse, a parabola, or a hyperbola. 
(e) This is a hyperboloid of 2 sheets. What is the intersection of the plane $x = 2020$ with the graph in the problem? Answer a point, two lines, an ellipse, a parabola, or a hyperbola. 	(f) This is a hyperbolic paraboloid. What is the intersection of the plane $y = 2020$ with the graph in the problem? Answer a point, two lines, an ellipse, a parabola, or a hyperbola.