Solutions to Spring, 2015, Math 241, Quiz 3

For the problems below, lines ℓ_1 and ℓ_2 are given by the following parametric equations.

$$\ell_1 : \begin{cases} x = & -t \\ y = & 1-t \\ z = & t \end{cases} \qquad \ell_2 : \begin{cases} x = & s \\ y = & 1-s \\ z = & 1-s \end{cases}$$

1. Explain why the lines do NOT intersect. Use complete English sentences and be precise.

   If t and s lead to a common point on ℓ_1 and ℓ_2 (i.e., an intersection point), then the three equations $-t = s$, $1 - t = 1 - s$ and $t = 1 - s$ will all be satisfied. Combining the first of these equations and the third of these equations, we obtain $-t + t = s + (1 - s)$ (the left-hand sides of the equations added together equal the right-hand side of the equations added together). This simplifies to $0 = 1$, which is incorrect. So ℓ_1 and ℓ_2 do not intersect.

   OR

   Equating the x, y and z values for ℓ_1 and ℓ_2, we get $-t = s$, $1 - t = 1 - s$ and $t = 1 - s$. The first equation implies $1 - t = 1 + s$ (since $s = -t$). The second equation now implies $1 + s = 1 - s$ which simplifies to $2s = 0$. We deduce that $s = 0$. Also, $t = 0$ (since $s = -t$). Plugging in $t = 0$ and $s = 0$ into the parametric equations for ℓ_1 and ℓ_2 give the points $(0, 1, 0)$ and $(0, 1, 1)$. These points are not equal, so ℓ_1 and ℓ_2 do not intersect.

2. Explain why the lines are NOT parallel. Use complete English sentences and be precise.

   The line ℓ_1 is parallel to the vector $\overrightarrow{v}_1 = \langle -1, -1, 1\rangle$, and the line ℓ_2 is parallel to the vector $\overrightarrow{v}_2 = \langle 1, -1, -1\rangle$. The lines ℓ_1 and ℓ_2 are not parallel because $\overrightarrow{v}_1$ is not some number k times $\overrightarrow{v}_2$. To see the latter, suppose $\overrightarrow{v}_1 = k\overrightarrow{v}_2$. Then comparing first components, we see that $k = -1$. But then $k\overrightarrow{v}_2 = -\overrightarrow{v}_2 = \langle -1, 1, 1\rangle \neq \overrightarrow{v}_1$. So no such k exists, and ℓ_1 and ℓ_2 are not parallel.

3. Calculate the distance between ℓ_1 and ℓ_2. Do not use a formula for this distance (i.e., one where you just plug in numbers to get the answer) unless you derive the formula. I want to see how you are using vectors to get the answer.

   Distance: $\boxed{1/\sqrt{2}}$

   Solution: Let $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$ be as in Problem 2. A vector perpendicular to both ℓ_1 and ℓ_2 is $\overrightarrow{v}_1 \times \overrightarrow{v}_2 = \langle 2, 0, 2\rangle$ (you should show the work for this). The point $P = (0, 1, 0)$ is on ℓ_1, and the point $Q = (0, 1, 1)$ is on ℓ_2. Therefore, the vector $\overrightarrow{PQ} = \langle 0, 0, 1\rangle$ is a vector going from a point on ℓ_1 to a point on ℓ_2. The answer is obtained by finding the length of the projection of $\overrightarrow{PQ}$ onto $\langle 2, 0, 2\rangle$, which is $\left|\langle 0, 0, 1\rangle \bullet \langle 2, 0, 2\rangle\right| / \|\langle 2, 0, 2\rangle\| = 2/\sqrt{8} = 1/\sqrt{2}$.