More Solutions to Past Problems

(Problems that have been bugging some of you)

1999 Final Exam, Part I, Problem 3 (b): The directional derivative is always maximized by going in the direction of the gradient. Since $\nabla f(x, y) = \langle 2xy, x^2 - 3y^2 + 2\rangle$, we get $\nabla f(2, -1) = \langle -4, 3\rangle$. The problem asked for a unit vector going in this direction. Since $\|\langle -4, 3\rangle\| = 5$, the answer is $\langle -4/5, 3/5\rangle$.

In general, $\nabla f(a, b)/\|\nabla f(a, b)\|$ is a unit vector going in the direction that maximizes the directional derivative of f at the point (a, b) (or at the point (a, b, c) if the value of $z = f(x, y)$ is given). Also, $-\nabla f(a, b)/\|\nabla f(a, b)\|$ is a unit vector going in the direction that minimizes the directional derivative of f at the point (a, b) (or at the point (a, b, c)).

1999 Final Exam, Part I, Problem 4: I am simply testing here if you know the definition of a partial derivative (or of a derivative). You should know that

$$\frac{\partial f(x, y)}{\partial x} = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h} \quad \text{and} \quad \frac{\partial f(x, y)}{\partial y} = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

This problem is asking about the first of these with $f(x, y) = (x + 2y)^{3/2}$. So you should calculate $\partial f/\partial x = (3/2)(x + 2y)^{1/2}$ and then plug in $(x, y) = (2, 1)$ to get $f_x(2, 1) = (3/2)4^{1/2} = 3$. You can do this problem using the formula for $\partial f(x, y)/\partial y$ above, but you have to be more careful (which is the reason for the remark concerning 6, but I won't go into the wrong way to get 6). In this case, note that $f(x, y + h/2) = (x + 2(y + h/2))^{3/2} = (x + 2y + h)^{3/2}$. So $h/2$ needs to be used instead of h. In other words, use that

$$\frac{\partial f}{\partial y} = \lim_{h/2\to 0} \frac{f(x, y + h/2) - f(x, y)}{h/2} = 2 \lim_{h\to 0} \frac{(x + 2y + h)^{3/2} - (x + 2y)^{3/2}}{h}.$$

The 2 appearing in the front is because of the $h/2$ that was in the denominator. Now, you have to divide by 2 to get this last limit. Since $\partial f/\partial y = 3(x + 2y)^{1/2}$ and $\partial f(2, 1)/\partial y = 3 \cdot 2 = 6$, we get the answer is $6/2 = 3$.

1999 Final Exam, Part I, Problem 6 (c): The region you should be integrating over is $1/8$th of a circle of radius $\sqrt{2}$ centered at $(0, 0)$, in the first quadrant, and above the line $y = x$. Convert the problem to polar coordinates. Note that the presence of $x^2 + y^2$ is in the problem and the region described in the problem both hint that polar coordinates is the way to go here. The answer should be

$$\int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} r^3 \cdot r \, dr \, d\theta = \int_{\pi/4}^{\pi/2} \frac{r^5}{5}\bigg|_0^{\sqrt{2}} d\theta = \frac{4\sqrt{2}}{5} \int_{\pi/4}^{\pi/2} d\theta = \frac{\pi\sqrt{2}}{5}.$$

1999 Final Exam, Part II, Problem 4: The graph of $z^2 = -1 + x^2 + y^2$ is an hyperboloid of one sheet that intersects the xy-plane (where $z = 0$) along the circle $x^2 + y^2 = 1$ and spreads out as the value of z increases (intersects planes parallel to the xy-plane in larger and larger circles as the planes go higher and higher above the xy-plane). The graph of $3x^2 + 3y^2 + z^2 = 4$ is an ellipsoid centered at the origin that intersects the xy-plane in the circle $x^2 + y^2 = 4/3$ (which has radius > 1). The solid is therefore inside the hyperboloid of one sheet, bounded below by $z = 0$ (this is given in the problem) and bounded above by the ellipsoid. Note that the teacher might give you a picture in a problem like this. However, since you are now being tested on the graphs of these solids from the first part of the course, it is reasonable to expect you to be able to reason as I have above.

Why do you need two integrals to set up this problem? Observe that part of the solid can be described as being inside the cylinder $x^2 + y^2 = 1$, bounded below by $z = 0$ and above by the ellipsoid $3x^2 + 3y^2 + z^2 = 4$. The other part is outside the cylinder $x^2 + y^2 = 1$, bounded below by the hyperboloid $z^2 = -1 + x^2 + y^2$ and above by the ellipsoid $3x^2 + 3y^2 + z^2 = 4$. Again, you should be able to see this without a picture given to you.

For the second part of this solid just described, you will want to know the region in the xy-plane that you are integrating over. It is apparently outside the circle $x^2 + y^2 = 1$ but inside something else. To figure out what the "inside something else" is, we need to see where the bottom of the solid and the top of the solid intersect. After multiplying $z^2 = -1 + x^2 + y^2$ by 3 and rearranging, the equations for the top and bottom of this part of the solid can be written as

$$3x^2 + 3y^2 = 3z^2 + 3 \quad \text{and} \quad 3x^2 + 3y^2 = 4 - z^2.$$

So the surfaces intersect where $3z^2 + 3 = 4 - z^2$, which gives $z = \pm 1/2$. Since the solid is above the xy-plane, we are interested in $z = 1/2$. Hence, the solid intersects where $1/4 = -1 + x^2 + y^2$ or $x^2 + y^2 = 5/4$. So the second part of the solid is above the region in the plane that is outside the circle $x^2 + y^2 = 1$ and inside the circle $x^2 + y^2 = 5/4$.

Use cylindrical coordinates to set up the integrals. The volume is

$$\int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-3r^2}} r\, dz\, dr\, d\theta + \int_0^{2\pi} \int_1^{\sqrt{5/4}} \int_{\sqrt{-1+r^2}}^{\sqrt{4-3r^2}} r\, dz\, dr\, d\theta.$$

The first integral can be evaluated as

$$\begin{aligned} \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-3r^2}} r\, dz\, dr\, d\theta &= \int_0^{2\pi} \int_0^1 r\sqrt{4-3r^2}\, dr\, d\theta \\ &= \int_0^{2\pi} -\frac{1}{9}(4-3r^2)^{3/2}\Big|_0^1\, d\theta = \frac{7}{9}\int_0^{2\pi} d\theta = \frac{14\pi}{9}. \end{aligned}$$

The second integral is

$$\begin{aligned} \int_0^{2\pi} \int_1^{\sqrt{5/4}} \int_{\sqrt{-1+r^2}}^{\sqrt{4-3r^2}} r\, dz\, dr\, d\theta &= \int_0^{2\pi} \int_1^{\sqrt{5/4}} (\sqrt{4-3r^2} - \sqrt{-1+r^2})\, r\, dr\, d\theta \\ &= \int_0^{2\pi} -\frac{1}{9}(4-3r^2)^{3/2} - \frac{1}{3}(-1+r^2)^{3/2}\Big|_1^{\sqrt{5/4}}\, d\theta \end{aligned}$$

$$= \int_0^{2\pi} \left(\frac{7}{72} - \frac{1}{24} \right) d\theta = \frac{1}{18} \int_0^{2\pi} d\theta = \frac{\pi}{9}.$$

The answer is $(14\pi)/9 + \pi/9 = 15\pi/9 = 5\pi/3$.

The problem could have been made a little easier (just a little) by writing the volume as a difference of two volumes. The easiest way is probably to use the set-up

$$\int_0^{2\pi} \int_0^{\sqrt{5/4}} \int_0^{\sqrt{4-3r^2}} r\, dz\, dr\, d\theta - \int_0^{2\pi} \int_1^{\sqrt{5/4}} \int_0^{\sqrt{-1+r^2}} r\, dz\, dr\, d\theta.$$

You may have to think about this approach to see it, but you should get the same answer if you do the calculation.