Calculus III: Homework Problem Sets
Part III

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§17. Homework Set 17: Definition of a Multiple Integral

FQ 1. Evaluate the double integral below by first identifying it as the volume of a solid.

$$\iint_R 2\,dA, \quad R = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le x\}$$

FQ 2. What is the value of the integral

$$\iint_R 2\,dA,$$

where $R = \{(x,y) : x^2 + y^2 \le 4\}$?

FF 3. Calculate $\iint_R dA$, where R is the region in the first quadrant bounded by the circle $x^2+y^2 = 1$ and the coordinate axes.

FQ 4. Calculate

$$\iint_R (5-x)\,dA, \qquad \text{where } R = \{(x,y) \,|\, 0 \le x \le 5, 0 \le y \le 3\},$$

by first interpreting the double integral as the volume of a solid.

FT 5. Define $R = \{(x,y) : 0 \le x \le 4, 0 \le y \le 5\}$ and

$$f(x,y) = \begin{cases} 2 & \text{if } 0 \le x \le 4 \text{ and } 0 \le y \le 3 \\ 1 & \text{if } 0 \le x \le 2 \text{ and } 3 < y \le 5 \\ -3 & \text{if } 2 < x \le 4 \text{ and } 3 < y \le 5. \end{cases}$$

(a) Evaluate $\displaystyle\iint_R f(x,y)\; dA$.

(b) Evaluate $\displaystyle\int_3^4 \int_1^5 f(x,y)\, dy\, dx$.

FT 6. Define $R = \{(x,y) : 0 \le x \le 3, 0 \le y \le 5\}$ and

$$f(x,y) = \begin{cases} -2 & \text{if } 0 \le x \le 3 \text{ and } 0 \le y \le 2 \\ 0 & \text{if } 0 \le x \le 2 \text{ and } 2 < y \le 5 \\ 5 & \text{if } 2 < x \le 3 \text{ and } 2 < y \le 5. \end{cases}$$

Evaluate $\displaystyle\iint_R f(x,y)\; dA$.

FT 7. Let

$$R = \{(x,y) : 0 \le x \le 4, 0 \le y \le 2\}$$

and

$$f(x,y) = \begin{cases} 2 & \text{for } 0 \le x \le 3 \text{ and } 0 \le y \le 1 \\ 1 & \text{for } 0 \le x \le 3 \text{ and } 1 < y \le 2 \\ -3 & \text{for } 3 < x \le 4 \text{ and } 0 \le y \le 2. \end{cases}$$

Calculate $\displaystyle\iint_R f(x,y)\, dA$.

FT 8. Let

$$R = \{(x,y) : 0 \le x \le 4, 0 \le y \le 3\}$$

and

$$f(x,y) = \begin{cases} 3 & \text{for } 0 \le x \le 3 \text{ and } 0 \le y \le 3 \\ -2 & \text{for } 3 < x \le 4 \text{ and } 0 \le y \le 2 \\ 1 & \text{for } 3 < x \le 4 \text{ and } 2 < y \le 3. \end{cases}$$

Calculate $\displaystyle\iint_R f(x,y)\, dA$.

FT 9. Calculate $\displaystyle\iint_R f(x,y)\, dA$, where

$$R = \{(x,y) : 0 \le x \le 3, 0 \le y \le 3\}$$

and

$$f(x,y) = \begin{cases} -2 & \text{for } 0 \le x \le 3 \text{ and } 0 \le y \le 1 \\ -1 & \text{for } 0 \le x \le 2 \text{ and } 1 < y \le 3 \\ 1 & \text{for } 2 < x \le 3 \text{ and } 1 < y \le 3. \end{cases}$$

FT 10. Calculate $\displaystyle\iint_R f(x,y)\,dA$, where

$$R = \{(x,y) : 0 \le x \le 4, 0 \le y \le 3\}$$

and

$$f(x,y) = \begin{cases} -2 & \text{for } 0 \le x \le 4 \text{ and } 0 \le y \le 2 \\ 3 & \text{for } 0 \le x \le 2 \text{ and } 2 < y \le 3 \\ 2 & \text{for } 2 < x \le 4 \text{ and } 2 < y \le 3. \end{cases}$$

FT 11. Let

$$R = \{(x,y) : 0 \le x \le 3, 0 \le y \le 4\}$$

and

$$f(x,y) = \begin{cases} 2 & \text{for } 0 \le x \le 2 \text{ and } 0 \le y \le 2 \\ -1 & \text{for } 0 \le x \le 2 \text{ and } 2 < y \le 4 \\ 1 & \text{for } 2 < x \le 3 \text{ and } 0 \le y \le 4. \end{cases}$$

(a) Calculate $\displaystyle\iint_R f(x,y)\,dA$.

(b) Calculate $\displaystyle\int_0^1 \int_0^3 f(x,y)\,dx\,dy$.

Answers for §17

1. 4
2. 8π
3. $\pi/4$
4. $75/2$
5. (a) 16
   (b) -2
6. 3

7. 3

8. 24

9. -8

10. -6

11. (a) 8
    (b) 5

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§18. Homework Set 18: Double Integrals over Rectangular Regions

FQ 1. Evaluate the iterated integral $\int_0^1 \int_0^2 (x+3)\,dy\,dx$.

FT 2. Calculate $\int_0^1 \int_0^2 xy\,dy\,dx$.

FQ 3. Calculate the value of the double integral

$$\int_0^2 \int_0^1 (2x+y)^4\,dx\,dy.$$

You do not need to simplify your answer (as long as it only involves numbers and arithmetic).

FQ 4. Calculate $\int_0^2 \int_{-4}^4 (xy+3)\,dx\,dy$. Simplify your answer.

FQ 5. Use a double integral to calculate the volume of the solid under the plane $z = 2x + y$ and over the rectangle

$$R = \{(x,y) : 3 \le x \le 5, 1 \le y \le 2\}.$$

FQ 6. Use a double integral to find the volume of the solid under the plane $z = 2x + 2y$ and over the rectangle

$$R = \{(x,y) : 2 \le x \le 3,\ 1 \le y \le 3\}.$$

FQ 7. Calculate $\int_1^2 \int_2^3 6xy^2\,dy\,dx$.

FT 8. Calculate $\int_0^\pi \int_0^1 dr\,d\theta$.

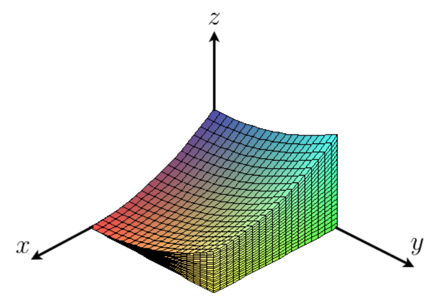
FT 9. Calculate $\displaystyle\int_{\pi/2}^{\pi}\int_{1}^{3}\cos\theta\,dr\,d\theta$.

FT 10. Calculate $\displaystyle\int_{0}^{\pi}\int_{0}^{2}\sin\theta\,dr\,d\theta$.

FT 11. Calculate $\displaystyle\int_{0}^{\pi/2}\int_{0}^{1}r\cos\theta\,dr\,d\theta$.

FQ 12. Calculate the value of $\displaystyle\int_{0}^{2}\int_{0}^{1}\big(2x+y\big)^{8}\,dx\,dy$.

FQ 13. Calculate the volume of the solid bounded between the surface $z = 15(x-1)^2+3y^2$ and the plane $z = 0$ and bounded on the sides by the planes $x = 0$, $x = 1$, $y = 0$ and $y = 2$. Simplify your answer.



(not scaled correctly)

FQ 14. Calculate the value of the double integral below.

$$\iint_R \big(3-2x\big)\,dA, \qquad \text{where } R = \{(x,y) \mid 0 \le x \le 3/2,\ 0 \le y \le 2\}.$$

Answers for §18

1. 7

2. 1

3. $\dfrac{4^6}{60} - \dfrac{2^7}{60} = \dfrac{992}{15}$

4. 48

5. 19

6. 18

7. 57

8. π

9. -2

10. 4

11. $1/2$

12. $\dfrac{1}{180}\left(4^{10}-2^{11}\right)=\dfrac{261632}{45}$

13. 18

14. $9/2$

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§19. Homework Set 19: Double Integrals over General Regions

FQ 1. Calculate $\displaystyle\int_0^1\int_0^{x^2} 3xy^2\,dy\,dx.$

FT 2. Calculate $\displaystyle\int_0^1\int_{x^2}^1 x^3\,dy\,dx.$

FF 3. Calculate $\displaystyle\int_0^1\int_0^x x^3y\,dy\,dx.$

FT 4. Calculate $\displaystyle\int_0^1\int_y^{y^2} y\,dx\,dy.$

FT 5. Calculate $\displaystyle\int_1^2\int_0^y xy\,dx\,dy.$

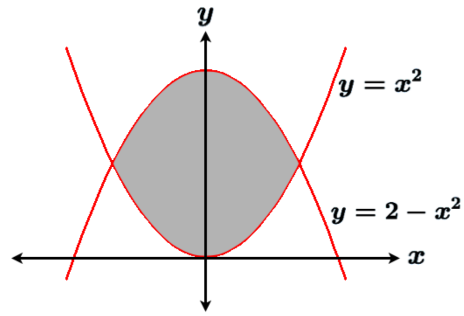
FQ 6. Calculate $\displaystyle\int_0^1\int_{-y}^1 20\,(x+y)^3\,dx\,dy.$

FT 7. Calculate $\displaystyle\int_0^\pi\int_0^\theta r\,dr\,d\theta.$

FQ 8. Find the volume of the solid that is under the plane $2y+z=2$ and above the region bounded by $y=x$ and $y=x^2$ in the xy-plane.

FQ 9. Let R be the region shown in the accompanying figure. Fill in the missing limits of integration.

$$\iint_R f(x,y)\,dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x,y)dydx$$



FQ 10. Fill in the five boxes below given that the double integral represents that volume of the solid enclosed by the parabolic cylinders $y = x^2$ and $y = 8 - x^2$ and the planes $z = 0$ and $3x + 4y - z = -9$. Do not calculate the double integral.

$$\int_{\square}^{\square} \int_{\square}^{\square} \boxed{}\; dy\,dx$$

11. Recall that the volume of a sphere of radius $r > 0$ is $(4/3)\pi r^3$. In the following, you should think in terms of the definition of a double integral. What is the value of each of the following integrals?

(a) $\displaystyle\int_0^r \int_0^{\sqrt{r^2-x^2}} \sqrt{r^2 - x^2 - y^2}\, dy\, dx$

(b) $\displaystyle\int_{-r}^r \int_0^{\sqrt{r^2-y^2}} \sqrt{r^2 - x^2 - y^2}\, dx\, dy$

(c) $\displaystyle\int_0^r \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \sqrt{r^2 - x^2 - y^2}\, dy\, dx$

(d) $\displaystyle\int_{-r}^r \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \sqrt{r^2 - x^2 - y^2}\, dx\, dy$

FQ 12. Fill in the five boxes below to correctly write down a double integral that represents the volume of the solid bounded by the coordinate planes and the plane $x + 4y + 2z = 4$.

$$\int_{\square}^{\square} \int_{\square}^{\square} \boxed{}\; dx\,dy$$

FQ 13. Calculate $\displaystyle\iint_R x^4\, dA$, where R is the region bounded below $y = 1 - x^2$ and above the x-axis.

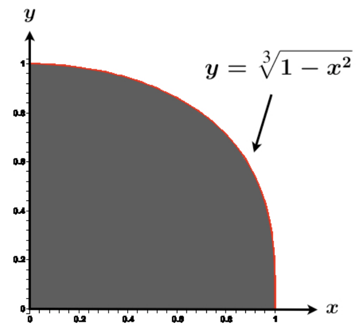
FT 14. Write each integral below as an iterated integral with the order of integration interchanged.

(a) $\displaystyle\int_0^1 \int_0^x f(x,y)\,dy\,dx$

(b) $\displaystyle\int_0^1 \int_{x^3}^1 f(x,y)\,dy\,dx$

FQ 15. Calculate the double integral below by first changing the order of integration.

$$\int_0^1 \int_0^{\sqrt[3]{1-x^2}} \sqrt{1-y^3}\;dy\,dx$$



FQ 16. Express the double integral below as an equivalent double integral with the order of integration reversed. Give precise limits of integration. (Note that there is no integral to evaluate here.)

$$\int_0^2 \int_{x^2}^4 f(x,y)\,dy\,dx$$

FQ 17. Reverse the order of integration, and calculate the value of

$$\int_0^{\sqrt{\pi/2}} \int_y^{\sqrt{\pi/2}} \cos(x^2)\,dx\,dy.$$

Simplify your answer.

FQ 18. Evaluate the double integral below by reversing the order of integration.

$$\int_0^1 \int_{\sqrt{y}}^1 \cos\left(x^3\right)\,dx\,dy$$

FQ 19. Evaluate the double integral by first reversing the order of integration.

$$\int_0^1 \int_{2x}^2 e^{-y^2}\,dy\,dx$$

FT 20. Interchange the order of integration to calculate the value of the double integral

$$\int_0^1 \int_0^{\sqrt{y}} (3x-x^3)^5\,dx\,dy.$$

FT 21. Interchange the order of integration and calculate the value of the double integral

$$\int_0^{\pi/2} \int_x^{\pi/2} \sqrt{\cos y}\;\cos x\,dy\,dx.$$

Simplify your answer.

FT 22. Interchange the order of integration and calculate the value of the double integral

$$\int_0^1 \int_{x^2}^1 e^{y^{3/2}}\, dy\, dx.$$

FT 23. Interchange the order of integration and calculate the value of the double integral

$$\int_0^1 \int_{\sqrt{y}}^1 \frac{x}{1+x^4}\, dx\, dy.$$

FT 24. Interchange the order of integration and calculate the value of the double integral

$$\int_0^1 \int_x^1 x\sqrt{1+y^3}\, dy\, dx.$$

Note that the answer should not "look" nice.

FQ 25. Switch (interchange) the order of integration below and give appropriate calculations to show that

$$\int_0^1 \int_y^1 \frac{y}{(x^3+1)^2}\, dx\, dy = \frac{1}{12}.$$

FQ 26. Calculate the value of the double integral below by reversing the order of integration.

$$\int_0^4 \int_{\sqrt{y}}^2 \frac{3}{x^3+1}\, dx\, dy$$

Your answer should involve the natural logarithm.

FQ 27. Calculate

$$\int_0^1 \int_y^{\sqrt{y}} \sin\left(\frac{x^2}{2} - \frac{x^3}{3}\right) dx\, dy.$$

Your answer should involve a trigonometric function.

FQ 28. Calculate the double integral below.

$$\int_0^2 \int_{y/2}^1 \sqrt{x^2+1}\, dx\, dy$$

You do not need to simplify your answer, but it should be a number.

FT 29. Calculate $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 e^{(x^3)}\, dx\, dy.$

FF 30. Calculate $\displaystyle\int_0^{\pi/2} \int_x^{\pi/2} \frac{\cos x}{\sqrt{1+\cos y}}\, dy\, dx.$

FF 31. Richard cannot remember how to calculate integrals involving inverse trigonometric functions, and his Calculus teacher asks him to calculate

$$\int_{1/2}^{1} \frac{\arcsin(\sqrt{x})}{\sqrt{x}}\, dx.$$

He has this crazy idea of looking instead at the double integral

$$\int_{1/2}^{1} \int_{\pi/4}^{\arcsin(\sqrt{x})} \frac{1}{\sqrt{x}}\, dy\, dx.$$

(a) Evaluate the "inner" integral (the one with respect to y) to get that the double integral is equal to a single integral with respect to x. What is the integral with respect to x that you get?

(b) Interchange the order of integration in the double integral above and evaluate it. (This is the important part of this problem. Calculate the double integral by switching the order of integration. Note that $y = \arcsin(\sqrt{x})$ is the same as $\sin y = \sqrt{x}$ for $x \in [1/2, 1]$ and $y \in [\pi/4, \pi/2]$.)

(c) Using the previous parts, justify that

$$\int_{1/2}^{1} \frac{\arcsin(\sqrt{x})}{\sqrt{x}}\, dx = \pi\left(1 - \frac{\sqrt{2}}{4}\right) - \sqrt{2}.$$

Answers for §19

1. 1/8
2. 1/12
3. 1/12
4. −1/12
5. 15/8
6. 31
7. $\pi^3/6$
8. 1/5
9. $\displaystyle\int_{-1}^{1} \int_{x^2}^{2-x^2} f(x,y)\, dy\, dx$

10. $\int_{-2}^{2}\int_{x^2}^{8-x^2}(9+3x+4y)\,dy\,dx$

11. (a) $\pi r^3/6$
    (b) $\pi r^3/3$
    (c) $\pi r^3/3$
    (d) $2\pi r^3/3$

12. $\int_0^1\int_0^{4-4y}\frac{4-x-4y}{2}\,dx\,dy$

13. 4/35

14. (a) $\int_0^1\int_y^1 f(x,y)\,dx\,dy$
    (b) $\int_0^1\int_0^{y^{1/3}} f(x,y)\,dx\,dy$

15. 3/4

16. $\int_0^4\int_0^{\sqrt{y}} f(x,y)\,dx\,dy$

17. 1/2

18. $(1/3)\sin(1)$

19. $(1/4)(1-e^{-4})$

20. 32/9

21. 2/3

22. $(2/3)(e-1)$

23. $(1/4)\ln(2)$

24. $(1/9)(2^{3/2}-1)$

25. $\int_0^1\int_0^x \frac{y}{(x^3+1)^2}\,dy\,dx = \int_0^1 \frac{x^2}{2(x^3+1)^2}\,dx = -\frac{1}{6}(x^3+1)^{-1}\Big|_0^1 = -\frac{1}{12}+\frac{1}{6}=\frac{1}{12}$

26. $\ln(9) = 2\ln(3)$

27. $1 - \cos(1/6)$

28. $(2/3)(2^{3/2} - 1)$

29. $(1/3)(e - 1)$

30. $2\sqrt{2} - 2$

31. (a) $\displaystyle\int_{1/2}^{1} \left(\frac{\arcsin(\sqrt{x})}{\sqrt{x}} - \frac{\pi/4}{\sqrt{x}} \right) dx$

 (b) $\dfrac{\pi}{2} - \sqrt{2}$

 (c) Since

$$\int_{1/2}^{1} \frac{\pi/4}{\sqrt{x}}\, dx = \frac{\pi\sqrt{x}}{2}\Bigg|_{1/2}^{1} = \frac{\pi}{2} - \frac{\pi\sqrt{2}}{4},$$

 we get from part (b) and then (a) that

$$\frac{\pi}{2} - \sqrt{2} = \int_{1/2}^{1}\int_{\pi/4}^{\arcsin(\sqrt{x})} \frac{1}{\sqrt{x}}\, dy\, dx = \int_{1/2}^{1} \left(\frac{\arcsin(\sqrt{x})}{\sqrt{x}} - \frac{\pi/4}{\sqrt{x}} \right) dx$$

$$= \int_{1/2}^{1} \frac{\arcsin(\sqrt{x})}{\sqrt{x}}\, dx - \int_{1/2}^{1} \frac{\pi/4}{\sqrt{x}}\, dx = \int_{1/2}^{1} \frac{\arcsin(\sqrt{x})}{\sqrt{x}}\, dx - \frac{\pi}{2} + \frac{\pi\sqrt{2}}{4}.$$

 Rearranging, we get

$$\int_{1/2}^{1} \frac{\arcsin(\sqrt{x})}{\sqrt{x}}\, dx = \frac{\pi}{2} - \sqrt{2} + \frac{\pi}{2} - \frac{\pi\sqrt{2}}{4} = \pi\left(1 - \frac{\sqrt{2}}{4}\right) - \sqrt{2},$$

 as we wanted.

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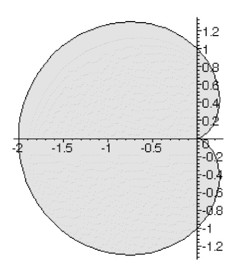
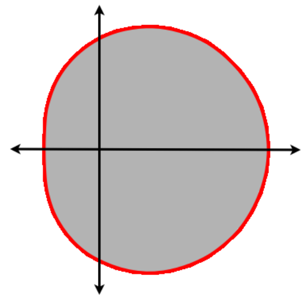
§20. Homework Set 20: Double Integrals using Polar Coordinates

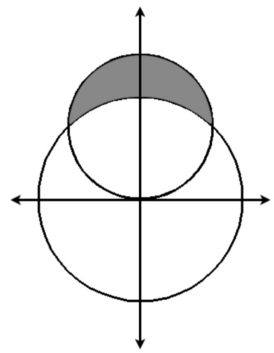
FF 1. Calculate the area of the region inside the curve $r = 1 - \cos\theta$ (see picture to the right). Simplify your answer.



FT 2. Calculate the area of the region enclosed by the graph of the limaçon $r = 2 + \cos\theta$.

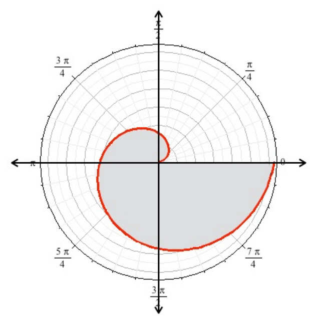


FQ 3. Let R be the region inside the circle $r = 2\sin\theta$ and outside the circle $r = \sqrt{2}$. Fill in the boxes below to indicate the correct limits of integration.

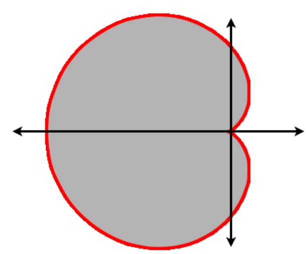


$$\iint_R f(r,\theta)\,dA = \int_{\square}^{\square}\int_{\square}^{\square} f(r,\theta)\,r\,dr\,d\theta$$

FQ 4. Calculate the area of the region enclosed by the graph of $r = \theta$ from $\theta = 0$ to $\theta = 2\pi$ and the positive x-axis (see the shaded region in the picture).

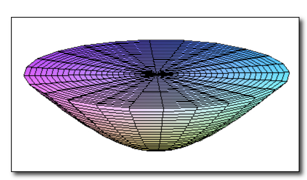


FQ 5. Use a double integral in polar coordinates to calculate the area of the region enclosed by the cardiod $r = 1 - \cos\theta$. Your final answer should be a number.



FQ 6. Calculate the area determined by one loop of the graph of $r = \cos(2\theta)$.

FQ 7. Fill in the five boxes below given that the double integral represents the volume of the solid enclosed by the hyperboloid $-x^2 - y^2 + z^2 = 1$ and the plane $z = 2$. Note that polar coordinates are being used and that I have only written $dr\,d\theta$ at the end so you should keep this in mind when feeling in the box immediately to the left of $dr\,d\theta$. Do not calculate the double integral.



$$\int_{\square}^{\square}\int_{\square}^{\square} \boxed{\qquad\qquad\qquad\qquad} \, dr\, d\theta$$

FQ 8. Calculate the value of

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \left(x^2+y^2\right)^{3/2} \, dx\, dy.$$

FQ 9. Calculate the double integral below by converting to polar coordinates.

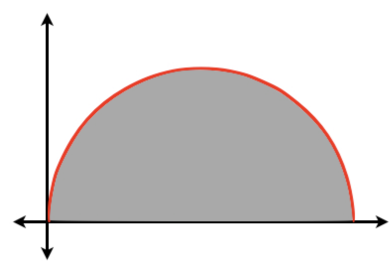
$$\int_0^3 \int_0^{\sqrt{9-x^2}} \left(25-(x^2+y^2)\right)^{1/2} \, dy\, dx$$

FQ 10. Evaluate the double integral

$$\int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} (x^2+y^2)^{1/2} \, dx\, dy$$

by converting to polar coordinates.

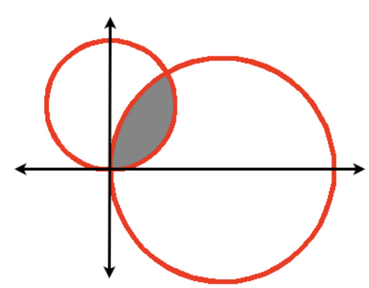
FQ 11. Calculate $\iint\limits_R \sin\theta \, dA$ where R is the region inside the graph of the half-circle $r = 2\cos\theta$ for $0 \le \theta \le \pi/2$ (pictured to the right). Note that this is not an area question.



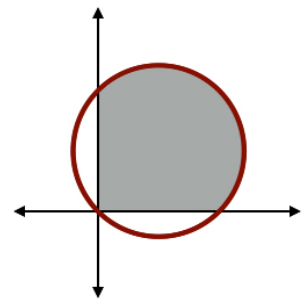
FQ 12. Calculate the volume of the solid which lies under the cone $z = \sqrt{x^2+y^2}$ and above the disk $x^2+y^2 \le 4$ in the xy-plane.

FQ 13. Express the volume of the solid that is inside $x^2+y^2 = 2y$ and lies above $z = 0$ and below $z = \sqrt{x^2+y^2}$ as a double integral in polar coordinates.

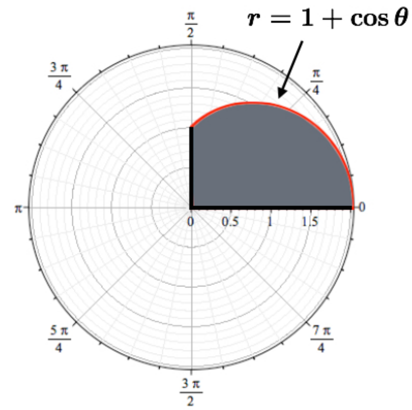
FT 14. The graph to the right shows the two circles $r = \sqrt{3}\cos\theta$ and $r = \sin\theta$. Express the area of the shaded region bounded by these two circles as a *sum of two* double integrals in polar coordinates. Do not evaluate the double integrals.



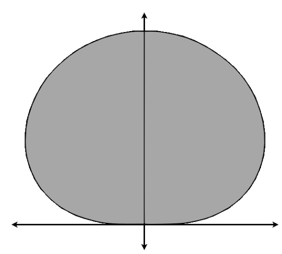
FT 15. The graph of $r = \cos\theta + \sin\theta$ is the circle pictured to the right. Calculate the shaded area in the picture, where $0 \leq \theta \leq \pi/2$. Your answer should be a number.



FT 16. The graph of $r = 1 + \cos\theta$ for $0 \leq \theta \leq \pi/2$ is pictured to the right. Calculate the shaded area in the picture to the right. Your answer should be a number.



FT 17. Calculate the area inside the graph of $r = \sqrt{\sin\theta}$ for $0 \leq \theta \leq \pi$ (pictured to the right).



Answers for §20

1. $3\pi/2$

2. $9\pi/2$

3. $\displaystyle\int_{\pi/4}^{3\pi/4} \int_{\sqrt{2}}^{2\sin\theta} f(r,\theta)\, r\, dr\, d\theta$

4. $4\pi^3/3$

5. $3\pi/2$

6. $\pi/8$

7. $\displaystyle\int_0^{2\pi}\int_0^{\sqrt{3}} \left(2-\sqrt{1+r^2}\right) r\,dr\,d\theta$

8. $\pi/10$

9. $61\pi/6$

10. 18π

11. $2/3$

12. $16\pi/3$

13. $\displaystyle\int_0^{\pi}\int_0^{2\sin\theta} r^2\,dr\,d\theta$

14. $\displaystyle\int_0^{\pi/3}\int_0^{\sin\theta} r\,dr\,d\theta + \int_{\pi/3}^{\pi/2}\int_0^{\sqrt{3}\cos\theta} r\,dr\,d\theta$

15. $\dfrac{\pi}{4}+\dfrac{1}{2}$

16. $\dfrac{3\pi}{8}+1$

17. 1

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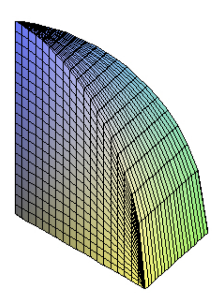
§21. Homework Set 21: Triple Integrals

FF 1. Calculate $\displaystyle\int_0^2\int_0^{3x}\int_y^{x+y} dz\,dy\,dx.$

FQ 2. Use a triple integral to calculate the volume of the wedge in the first octant that is cut from the solid cylinder $y^2+z^2\le 4$ by the planes $y=x$ and $x=0$.



FQ 3. Let G be the solid enclosed by the surfaces $z=4x^2+y^2$ and $z=4-3y^2$. Fill in the six missing limits of integration below.

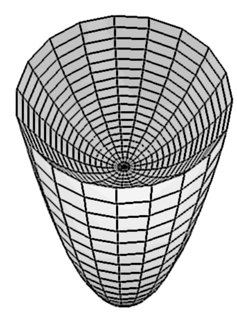
$$\iiint\limits_G f(x,y,z)\,dV = \int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} f(x,y,z)\,dz\,dy\,dx$$

FQ 4. The solid G is bounded below by $z = x^2 + 2y^2$, above by $z = 12 - 2x^2 - y^2$ and lies between the planes $y = 1$ and $y = 2$. Express the volume of G as a triple integral. Do not evaluate the triple integral - your final answer should be a triple integral with appropriate limits of integration.

FF 5. Express the volume of the solid in the first octant bounded by the coordinate planes, the surface $y^2 + z^2 = 4$ and the plane $x = 3$ as an iterated triple integral. Do not evaluate the integral.

FT 6. Express the volume of the solid in the first octant and inside the cylinders $y^2 + z^2 = 2$ and $x^2 + y^2 = 1$ as an iterated integral. Do not evaluate the integral.

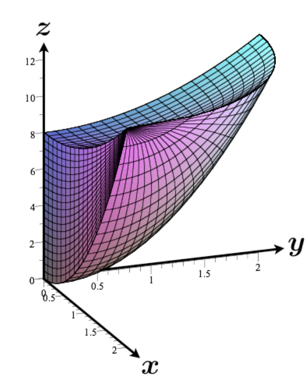
FT 7. Let G be the solid above the parabolic paraboloid $z = 3x^2 + 3y^2$ and below the parabolic paraboloid $z = 2 + x^2 + y^2$ (pictured to the right). Express the volume of G as a triple integral in rectangular coordinates x, y and z. Do not evaluate the triple integral.



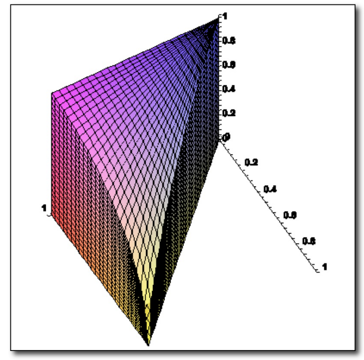
FT 8. Let G be the solid above the plane $z = 1$ and below the sphere $x^2 + y^2 + z^2 = 4$. Express the volume of G as a triple integral in rectangular coordinates x, y and z. Do not evaluate the triple integral.

FQ 9. Set up (but do not evaluate) an iterated triple integral for the volume of the solid inside the cylinder $x^2 + y^2 = 4$, above the plane $y + z = 0$ and below the plane $y - z = 0$. (Note: There are points (x, y, z) where $x < 0$ in the solid.)

FQ 10. Let G be the solid in the first octant (shown to the right) bounded by $x = 0$, $y = 0$, the paraboloid $z = 3x^2 + 3y^2$, and the paraboloid $z = x^2 + y^2 + 8$. Express the volume of G as a triple integral in rectangular coordinates x, y and z with appropriate limits of integration. Do not evaluate the triple integral.



FQ 11. Express the volume of the wedge in the first octant that is cut from the cylinder $y^2 + z^2 = 1$ by the planes $y = x$ and $x = 1$ as a triple integral. Answer by filling in each of the 5 boxes below with the appropriate number or expression. Do not evaluate the iterated integral.



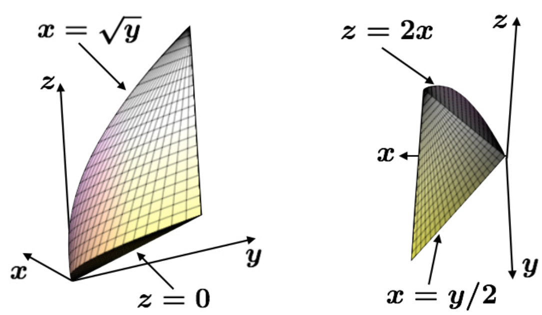
$$\int_0^1 \int_{\square}^{\square} \int_{\square}^{\square} \square \, dz\, dy\, dx$$

FQ 12. Write a triple integral in rectangular (Cartesian) coordinates (that is, in terms of x, y and z) that represents the volume of the solid bounded by $z = 3 - x^2 - y^2$ and $z = -6 + 2x^2 + 2y^2$. Do not evaluate the triple integral.

FQ 13. Rewrite the iterated integral below with the order of integration changed to "$dz\, dx\, dy$" and evaluate the integral with the changed order of integration.

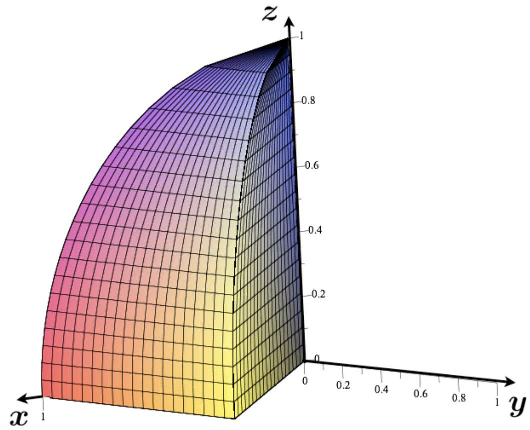
$$\int_0^1 \int_{x^2}^1 \int_0^y y^{-3/2} e^y dz\, dy\, dx$$

FQ 14. Fill in the six boxes below to correctly complete interchanging the order of integration. The first triple integral below should be used to obtain specific information about the solid. The images depict the solid that is to be used for the limits of integration. Here, the equations defining the faces are indicated by arrows; each arrow points to a face that is IN THE PICTURE.



$$\int_0^4 \int_{y/2}^{\sqrt{y}} \int_0^{2x} f(x,y,z)\, dz\, dx\, dy = \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} f(x,y,z)\, dy\, dx\, dz$$

FQ 15. Fill in the six boxes below to correctly complete interchanging the order of integration. The picture to the right depicts the solid that is to be used for the limits of integration, but the first triple integral below should be used to obtain specific information about the solid.



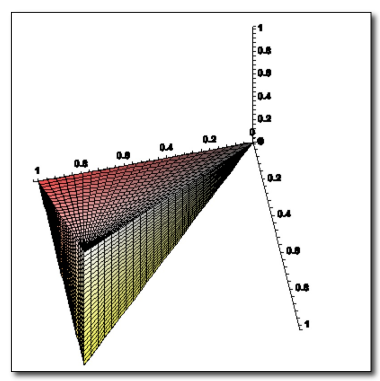
$$\int_0^1 \int_y^1 \int_0^{\sqrt{1-x^2}} f(x,y,z)\, dz\, dx\, dy$$

$$= \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} f(x,y,z)\, dx\, dy\, dz$$

FQ 16. Fill in the 6 boxes below to correctly complete interchanging the order of integration.

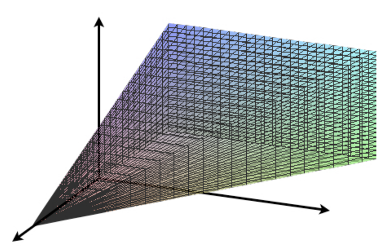
$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} f(x,y,z)\, dz\, dy\, dx = \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} f(x,y,z)\, dy\, dx\, dz$$

FQ 17. Write an iterated integral that is equal to the given iterated integral below but with the order of integration $dx\,dy\,dz$ instead of $dz\,dx\,dy$.



$$\int_0^1 \int_y^1 \int_0^y f(x,y,z)\, dz\, dx\, dy$$

FQ 18. Fill in the five boxes below to correctly complete interchanging the order of integration.



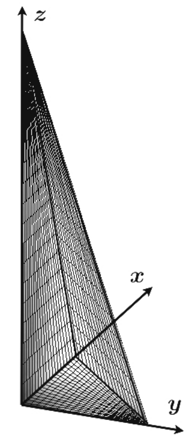
$$\int_{-3}^3 \int_0^{(6-2x)/3} \int_0^{(3-x)/3} f(x,y,z)\, dz\, dy\, dx$$

$$= \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} f(x,y,z)\, dy\, dx\, dz$$

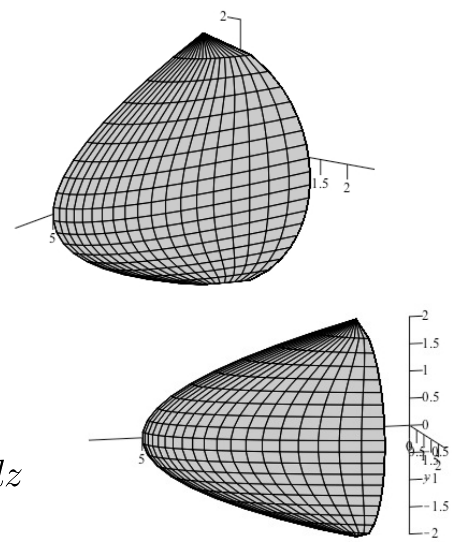
FT 19. Fill in the six boxes below to correctly complete interchanging the order of integration.

$$\int_0^6 \int_0^{(6-z)/3} \int_0^{(6-3y-z)/3} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} f(x,y,z)\,dz\,dy\,dx$$



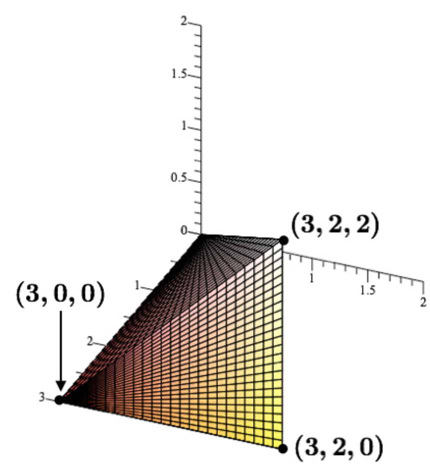
FT 20. Fill in the 6 boxes below to correctly complete interchanging the order of integration. The pictures to the right depict the solid that is to be used for the limits of integration, but the first triple integral below should be used to obtain specific information about the solid.



$$\int_{-2}^2 \int_0^{\sqrt{4-z^2}} \int_1^{5-y^2-z^2} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} f(x,y,z)\,dy\,dx\,dz$$

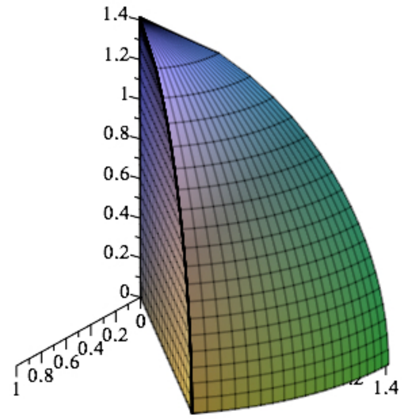
FT 21. Fill in the 6 boxes below to correctly complete interchanging the order of integration. The picture to the right depicts the solid that is to be used for the limits of integration, but the first triple integral below should be used to obtain specific information about the solid.



$$\int_0^3 \int_0^{2x/3} \int_0^y f(x,y,z)\,dz\,dy\,dx$$

$$= \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} f(x,y,z)\,dx\,dy\,dz$$

FT 22. Fill in the 6 boxes below to correctly complete interchanging the order of integration. The picture to the right depicts the solid that is to be used for the limits of integration, but the first triple integral below should be used to obtain specific information about the solid. Show work. In particular, you should not simply write down the upper limit of integration for x without showing where it came from.



$$\int_0^1 \int_x^{\sqrt{2-x^2}} \int_0^{\sqrt{2-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$

$$= \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} f(x,y,z)\,dy\,dx\,dz$$

FF 23. Calculate $\int_0^{\pi} \int_0^2 \int_0^1 zx^2 \sin(xyz)\,dx\,dy\,dz$. (Hint: If you do this problem using the approach intended, then the integration should not be hard.)

FF 24. Calculate $\int_0^{36} \int_{\sqrt{z}/2}^{3} \int_0^{3-y} \sqrt{4y^3 - y^4}\,dx\,dy\,dz$.

Answers for §21

1. 8

2. 8/3

3. $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{4x^2+y^2}^{4-3y^2} f(x,y,z)\ dz\,dy\,dx$

4. $\int_1^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{x^2+2y^2}^{12-2x^2-y^2} \ dz\,dx\,dy$

5. $\int_0^2 \int_0^{\sqrt{4-y^2}} \int_0^3 \ dx\,dz\,dy$ or $\int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^3 \ dx\,dy\,dz$

6. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{2-y^2}} \ dz\,dy\,dx$ or $\int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{\sqrt{2-y^2}} \ dz\,dx\,dy$

7. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{3x^2+3y^2}^{2+x^2+y^2} dz\,dy\,dx$ or $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{3x^2+3y^2}^{2+x^2+y^2} dz\,dx\,dy$

8. $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_{1}^{\sqrt{4-x^2-y^2}} dz\,dy\,dx$ or $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}}\int_{1}^{\sqrt{4-x^2-y^2}} dz\,dx\,dy$

9. $\displaystyle\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{-y}^{y} dz\,dy\,dx$

10. $\displaystyle\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{3x^2+3y^2}^{x^2+y^2+8} dz\,dy\,dx$

11. $\displaystyle\int_{0}^{1}\int_{0}^{x}\int_{0}^{\sqrt{1-y^2}} 1\ dz\,dy\,dx$

12. $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_{-6+2x^2+2y^2}^{3-x^2-y^2} dz\,dy\,dx$

13. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{y}}\int_{0}^{y} y^{-3/2}e^{y}\ dz\,dx\,dy = e-1$

14. $\displaystyle\int_{0}^{4}\int_{z/2}^{2}\int_{x^2}^{2x} f(x,y,z)\,dy\,dx\,dz$

15. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-z^2}}\int_{y}^{\sqrt{1-z^2}} f(x,y,z)\,dx\,dy\,dz$

16. $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-z^2}}\int_{0}^{\sqrt{9-x^2-z^2}} f(x,y,z)\,dy\,dx\,dz$

17. $\displaystyle\int_{0}^{1}\int_{z}^{1}\int_{y}^{1} f(x,y,z)\,dx\,dy\,dz$

18. $\displaystyle\int_{0}^{2}\int_{-3}^{3-3z}\int_{0}^{(6-2x)/3} f(x,y,z)\,dy\,dx\,dz$

19. $\displaystyle\int_{0}^{2}\int_{0}^{2-x}\int_{0}^{6-3x-3y} f(x,y,z)\,dz\,dy\,dx$

20. $\displaystyle\int_{-2}^{2}\int_{1}^{5-z^2}\int_{0}^{\sqrt{5-x-z^2}} f(x,y,z)\,dy\,dx\,dz$

21. $\displaystyle\int_{0}^{2}\int_{z}^{2}\int_{3y/2}^{3} f(x,y,z)\,dx\,dy\,dz$

22. $\displaystyle\int_{0}^{\sqrt{2}}\int_{0}^{\sqrt{(2-z^2)/2}}\int_{x}^{\sqrt{2-x^2-z^2}} f(x,y,z)\,dy\,dx\,dz$

23. $\pi/2$

24. $18\cdot 3^{3/2}$

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§22. Homework Set 22: Triple Integrals with Cylindrical Coordinates

FQ 1. Calculate the volume of the solid enclosed between the paraboloids $z = 3x^2 + 3y^2 - 7$ and $z = x^2 + y^2 + 1$.

FQ 2. Evaluate

$$\iiint_E \sqrt{x^2+y^2}\,dV,$$

where E is the solid that lies inside the cylinder $x^2 + y^2 = 9$ and between the planes $z = -1$ and $z = 1$.

FT 3. Express the volume of the solid in the first octant and inside the cylinders $y^2 + z^2 = 2$ and $x^2 + y^2 = 1$ as an iterated integral in cylindrical coordinates. Do not evaluate the integral.

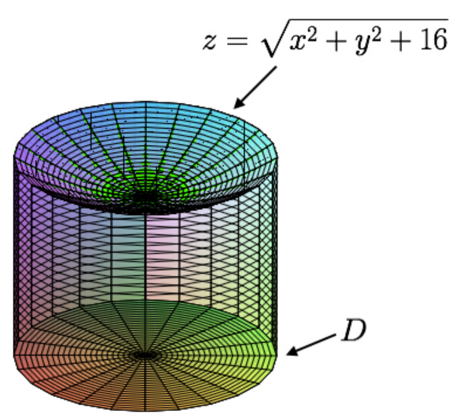
FQ 4. Calculate the value of

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{4} \left(x^2+y^2\right)^{3/2}\,dz\,dy\,dx.$$

FQ 5. Calculate the volume of the solid that is above the disk $D = \{(x,y)\,|\,x^2 + y^2 \le 9\}$ in the xy-plane and under the surface

$$z = \sqrt{x^2+y^2+16}.$$

Simplify your answer.



FQ 6. Find the volume of the solid that is bounded above and below by the sphere $x^2+y^2+z^2=9$ and inside the cylinder $x^2+y^2=8$. Simplify your answer.

FQ 7. Rewrite the triple integral below as a triple integral in cylindrical coordinates.

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-y^2} y^2\,dz\,dy\,dx$$

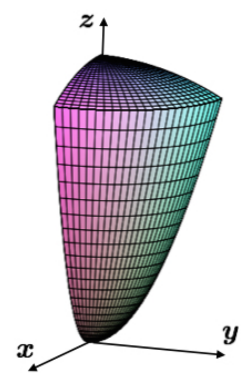
FT 8. Rewrite the triple integral below as a triple integral in cylindrical coordinates.

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} z\left(x^2+y^2+z^2\right)^{3/2}\,dz\,dy\,dx$$

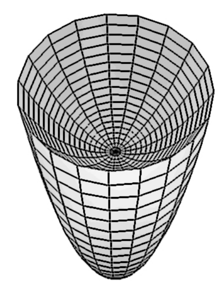
FT 9. Calculate the following triple integral by first rewriting the triple integral in cylindrical coordinates.

$$\int_{0}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{0}^{\sqrt{4-y^2-z^2}} \left(4-x^2-y^2\right)^2\,dx\,dz\,dy$$

FT 10. (5) Let G be the solid (shown to the right) in the first octant between the sphere $x^2+y^2+z^2=20$ and the paraboloid $z=x^2+y^2$. Express the volume of G as a triple integral in cylindrical coordinates. Do not evaluate the triple integral. (Hint: What is the value of z where the sphere and the paraboloid intersect?)



FT 11. Let G be the solid above the parabolic paraboloid $z=3x^2+3y^2$ and below the parabolic paraboloid $z=2+x^2+y^2$ (pictured to the right). Express the volume of G as a triple integral in cylindrical coordinates z, r and θ. Do not evaluate the triple integral.



FT 12. Let G be the solid between the two cylinders $4x^2+4y^2=\pi^2$ and $x^2+y^2=\pi^2$ and between the two planes $z=-1$ and $z=2$. Calculate

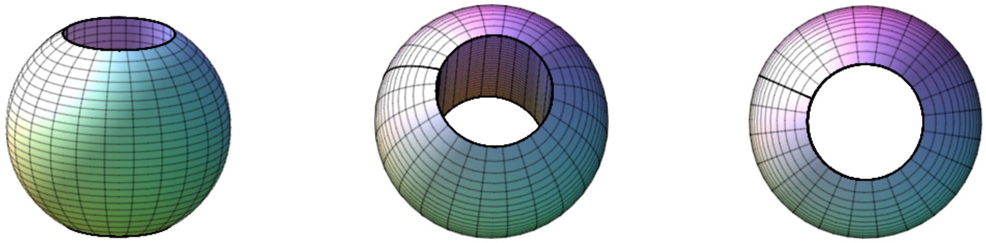
$$\iiint_G \frac{\cos\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}\,dV.$$

FF 13. Calculate $\iiint_S (x^2+y^2)^{3/2}\,dV$ where S is the solid inside the cylinder $x^2+y^2=1$, below the cone $z = 4 - \sqrt{x^2 + y^2}$ and above the plane $z = 0$.

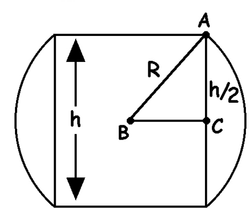
FQ 14. Evaluate the triple integral

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{9-x^2-y^2}} dz\, dy\, dx.$$

FF 15. This is a problem that is intended to have a surprising conclusion. It is a classical problem in Calculus (so you won't be surprised if you know the conclusion). We use material in this course to resolve it. Suppose we drill a cylindrical hole down the center of a sphere to make a bead (see the pictures below of different views of such a hole). The larger the sphere, the larger the bead will be. Suppose that the distance from the top of the sphere where the drill enters the sphere to the bottom of the sphere where the drill leaves the sphere is h. Let R denote the radius of the sphere.



(a) In the figure to the right, B is at the center of the sphere. What is the distance from B to C (the nearest point on the edge of the hole that is drilled out of the sphere)? Your answer should depend on R and h.



(b) Write a triple integral in cylindrical coordinates that represents the volume of the bead (formed from a sphere of radius R and with the "height" of the bead being h as described above).

(c) Calculate the volume of the bead using the triple integral in part (b).

(d) What happens to the volume of the bead as the radius R of the sphere increases?

Answers for §22

1. 16π

2. 36π

3. $\int_{0}^{\pi/2} \int_{0}^{1} \int_{0}^{\sqrt{2-r^2\sin^2\theta}} r\; dz\, dr\, d\theta$

4. $4\pi/5$

5. $122\pi/3$

6. $104\pi/3$

7. $\displaystyle\int_0^{\pi}\int_0^{2}\int_0^{4-r^2} r^3\sin\theta\ dz\,dr\,d\theta$

8. $\displaystyle\int_0^{\pi}\int_0^{1}\int_0^{\sqrt{1-r^2}} rz(r^2+z^2)^{3/2}\ dz\,dr\,d\theta$

9. $\displaystyle\int_0^{\pi/2}\int_0^{2}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} (4-r^2)^2\,r\ dz\,dr\,d\theta = \frac{128\pi}{7}$

10. $\displaystyle\int_0^{\pi/2}\int_0^{2}\int_{r^2}^{\sqrt{20-r^2}} r\ dz\,dr\,d\theta$

11. $\displaystyle\int_0^{2\pi}\int_0^{1}\int_{3r^2}^{2+r^2} r\ dz\,dr\,d\theta$

12. -6π

13. $19\pi/15$

14. $\dfrac{\pi}{3}\left(27-5^{3/2}\right)$

15. (a) $\sqrt{R^2-(h/2)^2}$

    (b) $\displaystyle\int_0^{2\pi}\int_{\sqrt{R^2-(h/2)^2}}^{R}\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r\ dz\,dr\,d\theta$

    (c) $\dfrac{4}{3}\pi\left(\dfrac{h}{2}\right)^3$

    (d) It's constant. The volume is always the same no matter what the value of R is.

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§23. Homework Set 23: Triple Integrals with Spherical Coordinates

FT 1. Calculate rectangular coordinates (x, y, z) and cylindrical coordinates (r, θ, z) for the point with spherical coordinates $(\rho, \theta, \phi) = (6, \pi/6, \pi/6)$. Simplify your answers so that no trigonometric and no inverse trigonometric functions are used.

FT 2. Calculate cylindrical coordinates (r, θ, z) and spherical coordinates (ρ, θ, ϕ) for the point with rectangular coordinates $(x, y, z) = (-\sqrt{3/2}, -1/\sqrt{2}, -\sqrt{2})$. Simplify your answers so that no trigonometric and no inverse trigonometric functions are used.

FT 3. Calculate cylindrical coordinates (r, θ, z) and spherical coordinates (ρ, θ, ϕ) for the point with rectangular coordinates $(x, y, z) = (\sqrt{3}, -1, -2)$. Simplify your answers so that no trigonometric and no inverse trigonometric functions are used.

FT 4. Calculate rectangular coordinates (x, y, z) and spherical coordinates (ρ, θ, ϕ) for the point with cylindrical coordinates $(r, \theta, z) = (2\sqrt{3}, -\pi/3, 6)$. Simplify your answers so that no trigonometric and no inverse trigonometric functions are used.

FT 5. Calculate the rectangular coordinates (x, y, z) and the spherical coordinates (ρ, θ, ϕ) for the point with cylindrical coordinates $(r, \theta, z) = (3, \pi/4, -\sqrt{3})$. Simplify your answers so that no trigonometric and no inverse trigonometric functions are used.

FF 6. Calculate cylindrical coordinates (r, θ, z) and spherical coordinates (ρ, θ, ϕ) for the point with rectangular coordinates $(x, y, z) = (3, 3\sqrt{3}, -2\sqrt{3})$. Simplify your answers so that no trigonometric and no inverse trigonometric functions are used.

FT 7. Calculate cylindrical coordinates (r, θ, z) and spherical coordinates (ρ, θ, ϕ) for the point with rectangular coordinates $(x, y, z) = (1, 1, \sqrt{6})$. Simplify your answers so that no trigonometric and no inverse trigonometric functions are used.

FT 8. Calculate rectangular coordinates (x, y, z) and cylindrical coordinates (r, θ, z) for the point with spherical coordinates $(\rho, \theta, \phi) = (4, 7\pi/6, \pi/6)$. Simplify your answers so that no trigonometric and no inverse trigonometric functions are used.

FT 9. Calculate cylindrical coordinates (r, θ, z) and spherical coordinates (ρ, θ, ϕ) for the point with rectangular coordinates $(x, y, z) = (-1, \sqrt{3}, -2\sqrt{3})$. Simplify your answers so that no trigonometric and no inverse trigonometric functions are used.

FT 10. Calculate the rectangular coordinates (x, y, z) and the cylindrical coordinates (r, θ, z) for the point with spherical coordinates $(\rho, \theta, \phi) = (4, \pi/2, \pi/6)$. Simplify your answers so that no trigonometric and no inverse trigonometric functions are used.

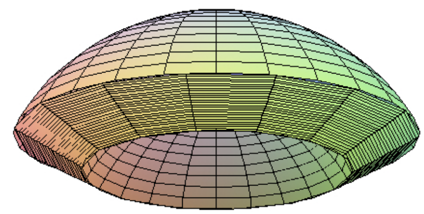
FT 11. Calculate rectangular coordinates (x, y, z) and spherical coordinates (ρ, θ, ϕ) for the point with cylindrical coordinates $(r, \theta, z) = (2, \pi/4, -2\sqrt{3})$. Simplify your answers so that no trigonometric and no inverse trigonometric functions are used.

FT 12. Complete the table below so that the variables x, y, z, r, θ, ρ and ϕ are chosen in such a way that the rectangular coordinates (x, y, z), cylindrical coordinates (r, θ, z) and spherical coordinates (ρ, θ, ϕ) all represent the same point. Simplify your answers so that they do not involve trigonometric functions and do not involve inverse trigonometric functions.

Variables	x	y	z	r	θ	ρ	ϕ
Values	3		-6		$11\pi/6$		

FF 13. Calculate $\displaystyle\int_0^{\pi}\int_0^{\pi/2}\int_0^{2} \rho\cos\theta\, d\rho\, d\theta\, d\phi$.

FQ 14. Let G be the solid inside the cone $z = \sqrt{(x^2+y^2)/3}$ and between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=9$ (as shown). Express the volume of G as a triple integral in spherical coordinates ρ, θ and ϕ with appropriate limits of integration. Do not evaluate the triple integral.

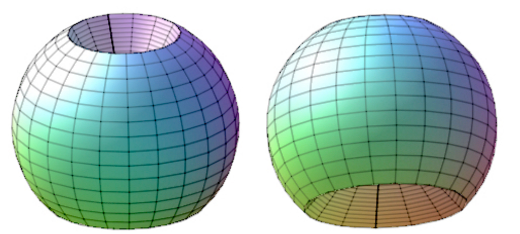


FQ 15. Use spherical coordinates to find the volume of the solid bounded above by the sphere $\rho = 3$ and below by the cone $\phi = \pi/6$.

FT 16. Let G be the solid above the plane $z = 1$ and below the sphere $x^2+y^2+z^2=4$. Express the volume of G as a triple integral in spherical coordinates ρ, ϕ and θ. Do not evaluate the triple integral.

FQ 17. Calculate the volume of the solid inside the half-cone $z = \sqrt{3x^2+3y^2}$ and between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$.

FQ 18. Express the volume of the solid that lies inside the sphere $x^2+y^2+z^2=9$ and between the two half-cones given by $z=\sqrt{3(x^2+y^2)}$ and $z=-\sqrt{x^2+y^2}$ as a triple integral in spherical coordinates. Do not use inverse trigonometric functions in your answer.



FQ 19. Express the volume of the solid that lies within the sphere $x^2+y^2+z^2=4$, above the xy-plane, and below the cone $z=\sqrt{x^2+y^2}$ as a triple integral in spherical coordinates.

FT 20. Rewrite the triple integral below as a triple integral in spherical coordinates.

$$\int_{-1}^{1}\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} z\left(x^2+y^2+z^2\right)^{3/2} dz\, dy\, dx$$

FQ 21. Express the volume of the solid that lies inside the sphere $x^2+y^2+z^2=4$ and between the two half-cones given by $z=\sqrt{(x^2+y^2)/3}$ and $z=-\sqrt{x^2+y^2}$ as a triple integral in spherical coordinates. Do not use inverse trigonometric functions in your answer.

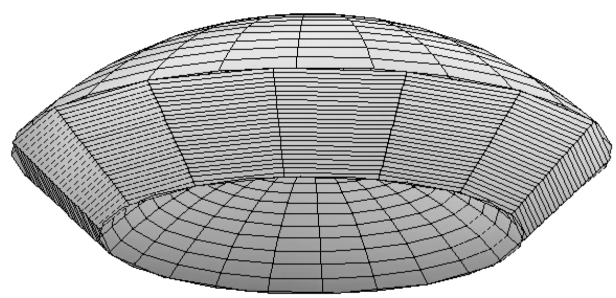
FT 22. Express the volume of the solid inside the sphere $x^2+y^2+z^2=4$ and between the two cones $z=\sqrt{x^2+y^2}$ and $z=-\sqrt{3(x^2+y^2)}$ as a triple integral in spherical coordinates. Do not use inverse trigonometric functions in your answer. Do not evaluate the triple integral.

FT 23. Express the volume of the solid within the sphere $x^2+y^2+z^2=8$ and *outside* the half-cone $z=\sqrt{x^2+y^2}$ as an iterated triple integral in spherical coordinates. Do not evaluate the triple integral.

FT 24. Write a triple integral in spherical coordinates which represents the volume of the solid above the surface $z=\sqrt{x^2+y^2}/\sqrt{3}$ and below the surface $x^2+y^2+z^2=4$.

FT 25. Write a triple integral in spherical coordinates which represents the volume of the solid inside the sphere $x^2+y^2+z^2=9$ and *above* the surface $z=-\sqrt{x^2+y^2}$. Do not evaluate the integral.

FT 26. Let G be the solid inside the cone $z=\sqrt{x^2+y^2}$ and between the spheres $x^2+y^2+z^2=2$ and $x^2+y^2+z^2=4$ (as shown to the right). Express the volume of G as a triple integral in spherical coordinates ρ, θ and ϕ with appropriate limits of integration. Do not evaluate the triple integral.



FQ 27. Fill in the 6 boxes below to correctly convert the integration using rectangular coordinates to an integration using spherical coordinates.

$$\int_0^3\int_0^{\sqrt{9-x^2}}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$

$$=\int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} f(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

FT 28. Let G be the solid between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=16$ and above the half-cone given by $z=-\sqrt{3\left(x^2+y^2\right)}$. Express the volume of G as a triple integral in spherical coordinates ρ, θ and ϕ with appropriate limits of integration. Simplify so that no inverse trigonometric functions appear in your answer. Do not evaluate the triple integral.

FT 29. Rewrite the following triple integral as a triple integral in spherical coordinates. Do not use inverse trigonometric functions in your answer. Do not evaluate the triple integral.

$$\int_0^3\int_0^{\sqrt{9-z^2}}\int_0^{\sqrt{9-y^2-z^2}} \left(x^2+y^2+z^2\right)^{3/2}\,dx\,dy\,dz$$

FQ 30. Calculate

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} e^{-(x^2+y^2+z^2)^{3/2}}\,dz\,dy\,dx.$$

Simplify your answer.

Answers for §23

1. $(x,y,z) = (3\sqrt{3}/2, 3/2, 3\sqrt{3})$ and $(r,\theta,z) = (3, \pi/6, 3\sqrt{3})$

2. $(r,\theta,z) = (\sqrt{2}, 7\pi/6, -\sqrt{2}))$ and $(\rho,\theta,\phi) = (2, 7\pi/6, 3\pi/4)$

3. $(r,\theta,z) = (2, 11\pi/6, -2)$ and $(\rho,\theta,\phi) = (2\sqrt{2}, 11\pi/6, 3\pi/4)$

4. $(x,y,z) = (\sqrt{3}, -3, 6)$ and $(\rho,\theta,\phi) = (4\sqrt{3}, 5\pi/3, \pi/6)$

5. $(x,y,z) = (3\sqrt{2}/2, 3\sqrt{2}/2, -\sqrt{3})$ and $(\rho,\theta,\phi) = (2\sqrt{3}, \pi/4, 2\pi/3)$

6. $(r,\theta,z) = (6, \pi/3, -2\sqrt{3})$ and $(\rho,\theta,\phi) = (4\sqrt{3}, \pi/3, 2\pi/3)$

7. $(r,\theta,z) = (\sqrt{2}, \pi/4, \sqrt{6})$ and $(\rho,\theta,\phi) = (2\sqrt{2}, \pi/4, \pi/6)$

8. $(x,y,z) = (-\sqrt{3}, -1, 2\sqrt{3})$ and $(r,\theta,z) = (2, 7\pi/6, 2\sqrt{3})$

9. $(r,\theta,z) = (2, 2\pi/3, -2\sqrt{3})$ and $(\rho,\theta,\phi) = (4, 2\pi/3, 5\pi/6)$

10. $(x,y,z) = (0, 2, 2\sqrt{3})$ and $(r,\theta,z) = (2, \pi/2, 2\sqrt{3})$

11. $(x,y,z) = (\sqrt{2}, \sqrt{2}, -2\sqrt{3})$ and $(\rho,\theta,\phi) = (4, \pi/4, 5\pi/6)$

12.

Variables	x	y	z	r	θ	ρ	ϕ
Values	3	$-\sqrt{3}$	-6	$2\sqrt{3}$	$11\pi/6$	$4\sqrt{3}$	$5\pi/6$

13. 2π

14. $\displaystyle\int_{0}^{2\pi}\int_{0}^{\pi/3}\int_{2}^{3} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$

15. $\displaystyle 18\pi\left(1 - \frac{\sqrt{3}}{2}\right)$

16. $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_{1/\cos\phi}^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

17. $\displaystyle\frac{14\pi}{3}\left(1-\frac{\sqrt{3}}{2}\right)$

18. $\displaystyle\int_0^{2\pi}\int_{\pi/6}^{3\pi/4}\int_0^{3}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

19. $\displaystyle\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

20. $\displaystyle\int_0^{\pi}\int_0^{\pi/2}\int_0^{1}\rho^6\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta$

21. $\displaystyle\int_0^{2\pi}\int_{\pi/3}^{3\pi/4}\int_0^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

22. $\displaystyle\int_0^{2\pi}\int_{\pi/4}^{5\pi/6}\int_0^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

23. $\displaystyle\int_0^{2\pi}\int_{\pi/4}^{\pi}\int_0^{2\sqrt{2}}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

24. $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_0^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

25. $\displaystyle\int_0^{2\pi}\int_0^{3\pi/4}\int_0^{3}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

26. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_{\sqrt{2}}^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

27. $\displaystyle\int_0^{\pi/2}\int_0^{\pi}\int_0^{3}f(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

28. $\displaystyle\int_0^{2\pi}\int_0^{5\pi/6}\int_2^{4}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

29. $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{3}\rho^5\sin\phi\,d\rho\,d\phi\,d\theta$

30. $\frac{\pi}{3}\left(1-e^{-8}\right)$

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§24. Homework Set 24: Integrals - Various Approaches

FT 1. Calculate $\displaystyle\int_0^6 \int_{y/3}^2 y\sqrt{x^3+1}\, dx\, dy$.

FT 2. Calculate $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} y^{1/2}(1-x^2)^{1/4}\, dx\, dy$.

FT 3. Calculate $\displaystyle\int_0^1 \int_x^{\sqrt{2-x^2}} \left(x^2+y^2\right)^{3/2}\, dy\, dx$.

FT 4. Calculate $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}-x} x\cos\left(\left(\frac{\pi}{2}-y\right)^3\right) dy\, dx$.

FF 5. Calculate $\displaystyle\int_{-3}^3 \int_0^{\sqrt{9-y^2}} \frac{1}{\sqrt{7+x^2+y^2}}\, dx\, dy$.

FT 6. Calculate

$$\int_0^{2\sqrt{\pi-1}} \int_{y/2}^{\sqrt{\pi-1}} \sin\left(x^2+1\right) dx\, dy.$$

Your answer should be a number but may involve trigonometric functions.

FF 7. Calculate $\displaystyle\int_{-2}^2 \int_0^{\sqrt{4-y^2}} \left(x^2+y^2\right)^{1/2}\, dx\, dy$.

FF 8. Calculate $\displaystyle\int_{-1}^1 \int_{x^2}^1 \sin(y^{3/2})\, dy\, dx$.

FT 9. Calculate the value of the double integral $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 (3x^3+1)^{1/2}\, dx\, dy$.

FF 10. Calculate $\displaystyle\int_0^4 \int_3^{\sqrt{25-y^2}} \sqrt{25-x^2}\, dx\, dy$.

FT 11. Calculate $\displaystyle\int_{-4}^4 \int_0^{\sqrt{16-y^2}} e^{(x^2+y^2)}\, dx\, dy$.

FT 12. Calculate $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} z^2 \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx.$

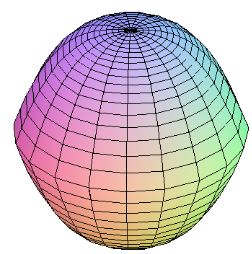
FF 13. Calculate $\displaystyle\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz\, dx\, dy.$

FT 14. Calculate the value of the triple integral $\displaystyle\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \int_0^y (x^2+y^2)^{3/2}\, dz\, dy\, dx.$

FF 15. Calculate $\displaystyle\int_{-4}^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-y^2}} \sqrt{1+(x^2+y^2+z^2)^{3/2}}\, dz\, dy\, dx.$

FT 16. Calculate the volume of the solid above the surface $z = 3x^2 + 3y^2$ and below the surface $z = 4 - x^2 - y^2$. Simplify your answer.

FT 17. Calculate the volume of the solid that lies between the surfaces $z = (x^2+y^2)^{3/2}$ and $z = 16 - (x^2+y^2)^{3/2}$.



FT 18. Calculate $\displaystyle\int_{-1}^3 \int_0^{\sqrt{\pi}} \int_0^{\sqrt{\pi-y^2}} \sin\left(x^2+y^2\right) dx\, dy\, dz.$

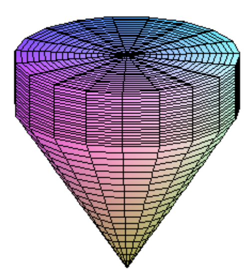
FT 19. Calculate

$$\int_0^4 \int_0^{\sqrt{16-z^2}} \int_{-\sqrt{16-y^2-z^2}}^{\sqrt{16-y^2-z^2}} \left(x^2+y^2+z^2\right)^{7/2} dx\, dy\, dz.$$

FT 20. Calculate

$$\iiint\limits_G \frac{(x^2+y^2)^{3/2}}{4-3\sqrt{x^2+y^2}}\, dV,$$

where G is the solid that lies above the cone $z = 3\sqrt{x^2+y^2}$, below the plane $z = 4$ and inside the cylinder $x^2+y^2 = 1$.



FT 21. Calculate

$$\iiint\limits_G \frac{1}{\sqrt{x^2+y^2+z^2}}\, dV,$$

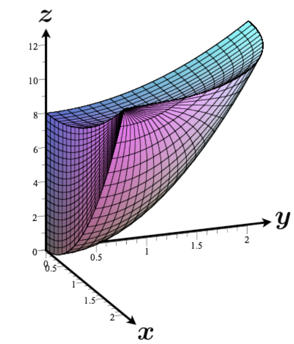
where G is the solid inside the cone $z = \sqrt{3(x^2+y^2)}$ and between the spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 9$.

FT 22. Let G be the solid inside the half-cone $3z = \sqrt{3x^2 + 3y^2}$ and between the spheres $x^2+y^2+z^2 = 1$ and $x^2 + y^2 + z^2 = 4$. Calculate

$$\iiint_G \frac{1}{x^2 + y^2 + z^2}\, dV.$$

FT 23. Let G be the solid in the first octant bounded by $x = 0$, $y = 0$, the paraboloid $z = 3x^2 + 3y^2$, and the paraboloid $z = x^2 + y^2 + 8$. Calculate

$$\iiint_G (x^2 + y^2)\, dV.$$



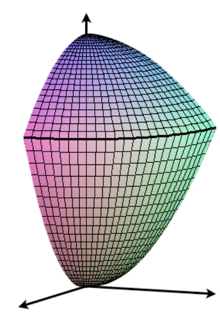
FQ 24. Let S be the solid in the first octant inside the cylinder $x^2+y^2 = 4$ and below the paraboloid $z = x^2 + y^2$. Calculate the value of

$$\iiint_S \frac{1}{\left(1 + (x^2 + y^2)^2\right)^2}\, dV.$$

FT 25. Calculate the value of

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{2x^2+2y^2}^{3-x^2-y^2} \sqrt{1 - x^2 - y^2}\, dz\, dy\, dx.$$

Simplify your answer. (The triple integral is over the solid shown to the right, but note that the problem is not asking for the volume of the solid.)



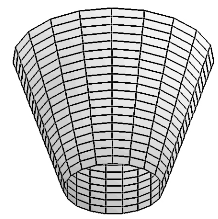
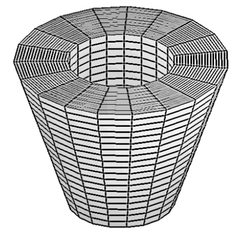
FT 26. Calculate

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{16-(x^2+y^2)}} (x^2 + y^2 + z^2)^{3/2}\, dz\, dy\, dx.$$

Your answer should be a number that does not involve trigonometric nor inverse trigonometric functions. But you do not need to simplify it in other ways. For example, you may want to use 2 or 4 to a power without working out what it equals.

FF 27. Calculate the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 50$ and below by the cone $z = 7\sqrt{x^2 + y^2}$. Justify your answer and simplify it so that it does not involve any trigonometric or inverse trigonometric functions.

FT 28. Calculate the volume of the solid inside the half-cone $z = 3\sqrt{x^2 + y^2}$, outside the cylinder $x^2 + y^2 = 1$, and between the planes $z = 3$ and $z = 6$. Two pictures of the solid from different angles are shown below.

FF 29. Calculate $\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{\sqrt{2x^2+2y^2}}^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx$. Simplify so that your final answer does not involve trigonometric functions.

FF 30. Calculate $\int_0^{\pi} \int_0^{2} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^2 \big(z^2+r^2\big)^{1/2}\, dz\, dr\, d\theta$.

FF 31. Calculate

$$\int_0^{\pi} \int_0^{2} \int_r^{\sqrt{8-r^2}} (r^2+z^2)^{1/2} r\, dz\, dr\, d\theta.$$

Simplify your answer. (Hint: Consider changing to a different coordinate system.)

FF 32. Calculate the value of the integral

$$\int_0^{2\pi} \int_0^{1} \int_{\sqrt{5}\,r}^{\sqrt{6-r^2}} (r^2+z^2)^{1/2} r\, dz\, dr\, d\theta.$$

Simplify your answer.

FF 33. Calculate the value of the integral

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^{2} \rho^3 \sin^2\!\phi \sqrt{4-\rho^2\sin^2\!\phi}\; d\rho\, d\phi\, d\theta.$$

Simplify your answer. (Hint: Consider changing to a different coordinate system.)

FF 34. Calculate the volume of the solid lying above the xy-plane and inside the surfaces

$$z^2 = -1 + x^2 + y^2 \qquad \text{and} \qquad 3x^2 + 3y^2 + z^2 = 4.$$

Simplify your answer. (Hint: Express the volume as a sum of two integrals.)

FT 35. Use a Jacobian (as indicated below) to calculate the value of

$$\iint_R (2x+y)^2(x-2y)\, dx\, dy, \qquad \text{where } R = \{(x,y) : 0 \le 2x+y \le 3, 1 \le x-2y \le 2\}.$$

To evaluate the integral, use the substitutions

$$u = 2x + y \quad \text{and} \quad v = x - 2y.$$

Your teacher has solved for x and y for you, so you don't need to. Solving for x and y gives

$$x = \frac{2u + v}{5} \quad \text{and} \quad y = \frac{u - 2v}{5}.$$

Now, calculate the Jacobian $\partial(x, y)/\partial(u, v)$ and the double integral above.

FT 36. Use a Jacobian (as indicated below) to calculate the value of

$$\iint_R (2x + 3y)^3 (x - 2y)^4 \, dx \, dy, \qquad \text{where } R = \{(x, y) : 0 \le 2x + 3y \le 2, 0 \le x - 2y \le 1\}.$$

To evaluate the integral, use the substitutions

$$u = 2x + 3y \quad \text{and} \quad v = x - 2y.$$

Your teacher has solved for x and y for you, so you don't need to. Solving for x and y gives

$$x = \frac{1}{7}(2u + 3v) \quad \text{and} \quad y = \frac{1}{7}(u - 2v).$$

Now, calculate the Jacobian $\partial(x, y)/\partial(u, v)$ and the double integral above.

Answers for §24

1. 26
2. $4/9$
3. $\pi\sqrt{2}/5$
4. $\sin\big((\pi/2)^3\big)/6$
5. $(4 - \sqrt{7}\,)\pi$
6. $1 + \cos(1)$
7. $8\pi/3$
8. $\frac{4}{3}\big(1 - \cos(1)\big)$
9. $14/27$
10. $52/3$

11. $\frac{\pi}{2}\left(e^{16}-1\right)$

12. $32\pi/9$

13. $16\pi/3$

14. $1/3$

15. $\frac{2\pi}{9}\left(65^{3/2}-1\right)$

16. 2π

17. $192\pi/5$

18. 2π

19. $\frac{4^{10}\pi}{10}=\frac{2^{19}\pi}{5}$

20. $2\pi/5$

21. $8\pi\left(1-\frac{\sqrt{3}}{2}\right)$

22. π

23. $16\pi/3$

24. $2\pi/17$

25. $3\pi/10$

26. $\frac{1024\pi}{3}\left(1-\frac{\sqrt{3}}{2}\right)$

27. $\frac{100\pi}{3}\left(\sqrt{50}-7\right)$

28. 4π

29. $\frac{81\pi}{2}\left(1-\sqrt{\frac{2}{3}}\right)$

30. $16\pi^2/5$

31. $16\pi\left(1-\dfrac{\sqrt{2}}{2}\right)$

32. $18\pi\left(1-\sqrt{\dfrac{5}{6}}\right)$

33. $128\pi/15$

34. $5\pi/3$

35. $\partial(x,y)/\partial(u,v) = -1/5$ and $\displaystyle\iint_R (2x+y)^2(x-2y)\,dx\,dy = \frac{27}{10}$

36. $\partial(x,y)/\partial(u,v) = -1/7$ and $\displaystyle\iint_R (2x+3y)^3(x-2y)^4\,dx\,dy = \frac{4}{35}$

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§25. Homework Set 25: Line Integrals

FF 1. Calculate the line integral $\displaystyle\int_{\mathcal{C}} (x+2y)\ dx + (x-2y)\ dy$ where $\mathcal{C}$ is the line segment from $(1,1)$ to $(3,-1)$.

FF 2. (a) Calculate the line integral $\displaystyle\int_{\mathcal{C}} (x^2\cos y + y)\ dx\ +\ (x^2\cos y - y)\ dy$ where $\mathcal{C}$ is the line segment from $(1,0)$ to $(0,1)$.

(b) What is the value of the line integral in part (a) if instead $\mathcal{C}$ is the line segment from $(0,1)$ to $(1,0)$?

FF 3. Calculate the line integral $\displaystyle\int_{\mathcal{C}} (x-2y)\,dx + (2x+y)\,dy$ where $\mathcal{C}$ is the semi-circle of radius 2 centered at the origin above and on the x-axis starting from $(2,0)$ and ending at $(-2,0)$.

FF 4. Calculate $\displaystyle\int_{\mathcal{C}} 2y\,dx + x\,dy + 2(y-z)\,dz$ where $\mathcal{C}$ is the curve given by $x = 1-t$, $y = t^2$, and $z = t^2 - t$ from $t = 0$ to $t = 1$.

Answers for §25

1. 0

2. (a) -1

   (b) 1

3. 8π

4. 0

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§26. Homework Set 26: Green's Theorem

FF 1. Using Green's Theorem, calculate the line integral

$$\int_{\mathcal{C}} (y^2 + x^3 - 2x)\,dx \;+\; (2xy + x^2 - 3\cos(y^2+1))\,dy,$$

where $\mathcal{C}$ is the triangle oriented counter-clockwise with vertices at $(0,0)$, $(2,2)$, and $(0,1)$.

FF 2. Use Green's Theorem to calculate $\int_{\mathcal{C}} (y+\sin x)\,dx + (3x - y^3\cos y)\,dy$, where $\mathcal{C}$ is the counter-clockwise oriented curve consisting of the line segment from $(0,0)$ to $(2,2)$, the portion of the circle $x^2+y^2=8$ from $(2,2)$ to $(-2,-2)$, and the line segment from $(-2,-2)$ to $(0,0)$.

FF 3. Using Green's Theorem, calculate the line integral

$$\int_{\mathcal{C}} \big(\cos x + \sin y - xy^3\big)\,dx + \big(x\cos y - x^2y^2 + \mathrm{e}^{y^2+1}\big)\,dy$$

where $\mathcal{C}$ is the rectangle oriented counter-clockwise with vertices $(0,0)$, $(1,0)$, $(1,3)$, and $(0,3)$. Simplify your answer.

FF 4. Using Green's Theorem, calculate

$$\int_{\mathcal{C}} (-3y + 2x\sin y)\,dx + (x + x^2\cos y)\,dy,$$

where $\mathcal{C}$ is the curve that is oriented counter-clockwise given by $x = \cos t$ and $y = \sin t$ for $0 \le t \le 2\pi$.

FF 5. The curve $\mathcal{C}$ consists of the segment from $(0,0)$ to $(4,0)$, the segment from $(4,0)$ to $(4,2)$, the segment from $(4,2)$ to $(0,2)$, and the segment from $(0,2)$ to $(0,0)$. Thus, $\mathcal{C}$ is the curve counter-clockwise traversing the edges of the rectangle with vertices $(0,0)$, $(4,0)$, $(4,2)$ and $(0,2)$. Using Green's theorem, calculate the value of the line integral

$$\int_{\mathcal{C}} xy^3\,dx + x^2y^2\,dy.$$

FF 6. Use Green's Theorem to calculate

$$\int_{\mathcal{C}} (y - y\, e^x + 2x \cos y)\, dx + (3x - e^x - x^2 \sin y)\, dy,$$

where $\mathcal{C}$ is the counter-clockwise oriented curve consisting of the line segment from $(-2, 0)$ to $(2, 0)$ and the portion of the circle $x^2 + y^2 = 4$ above the x-axis beginning at $(2, 0)$ and ending at $(-2, 0)$.

FF 7. What is the value of

$$\int_{\mathcal{C}} \left(e^{\sqrt{3x^2+2}} - y\right) dx + \Big(\big(\cos\left(2y^2 + 3\right) \big)^4 + x \Big)\, dy,$$

where $\mathcal{C}$ is the curve given by

$$x = \sin t, \quad y = \cos t, \quad 0 \le t \le 2\pi\,?$$

Answers for §26

1. $4/3$

2. 8π

3. $9/2$

4. 4π

5. $-64/3$

6. 4π

7. -2π (Note that $\mathcal{C}$ is oriented clockwise.)