Calculus III: Homework Problem Sets
Part II

§8. Homework Set 8: Vector Valued Functions - Basics

FT 1. Fill in the boxes here to make a correct statement. Use one of the choices indicated below the boxes. The graph described by the parametric equations

$$x = 4\cos t, \quad y = 3\sin t, \quad z = 5, \qquad \text{with } 0 \le t \le 2\pi$$

is $\boxed{}$ that is in a plane parallel to the $\boxed{}$.
("an ellipse", "a parabola" or "a line") ("xy-plane", "xz-plane" or "yz-plane")

FT 2. Describe the graph of $\overrightarrow{r}(t) = \langle \cos t, -2\sin t, 3\rangle$ by filling in the boxes appropriately. The graph is $\boxed{}$ that lies in the plane $\boxed{}$.
(a circle, an ellipse, a spiral, or a parabola) (an equation of a plane)

FQ 3. Two particles travel along the space curves

$$\overrightarrow{r}_1 = \langle t, t^2, t^3\rangle \quad \text{and} \quad \overrightarrow{r}_2 = \langle 1+2t, 1+6t, 1+14t\rangle.$$

Their paths intersect in two points. What are they?

FQ 4. (a) Two particles travel along the curves given by

$$\overrightarrow{r_1}(t) = \langle\, t^2,\, t,\, t^2+t\,\rangle \quad \text{and} \quad \overrightarrow{r_2}(t) = \langle\, t^2-t,\, 2t-2,\, t^2+t-2\,\rangle.$$

There are two points where their "paths" intersect. What are they?

(b) Do the particles in part (a) collide?

5. Two particles travel along the curves given by

$$\overrightarrow{r_1}(t) = \langle\, t^2 - 2t,\, t^3,\, 2t+1 \,\rangle \quad \text{and} \quad \overrightarrow{r_2}(t) = \langle\, t,\, t^2 + 6t,\, t+4 \,\rangle.$$

Do the particles collide? Explain your answer.

FQ 6. (a) At what value of $t \in [0, 2\pi)$ does the curve $x = 3\sin t$, $y = 4 - 4\sin t$ and $z = 5\cos t$ pass through the point $(0, 4, -5)$?

(b) Using your answer to (a), write parametric equations for the line tangent to the curve $x = 3\sin t$, $y = 4 - 4\sin t$ and $z = 5\cos t$ at the point $(0, 4, -5)$.

FQ 7. Find parametric equations of the line tangent to the graph of

$$\vec{r}(t) = \langle \sin t, \cos t, 2t \rangle$$

at the point where $t = 0$.

FQ 8. Calculate parametric equations for the tangent line to the curve given below at the indicated point P.

$$x = t^2, \quad y = 2t - 1, \quad z = t^2 - t, \quad P = (1, -3, 2)$$

FQ 9. Find parametric equations for the tangent line to the curve given by

$$x = 1 - t^2, \quad y = 1 - t^3, \quad z = e^{t+1}$$

at the point $(0, 2, 1)$.

FT 10. Find parametric equations for the line ℓ that is tangent to the curve $\vec{r}(t) = \langle t^2, t, t^3 \rangle$ at the point $(1, 1, 1)$.

FF 11. The two surfaces $z = x^2 - y^2 + 8$ and $z = 3x^2 + y^2$ intersect in a curve. The point $P = (\sqrt{2}, \sqrt{2}, 8)$ is on the curve. What are the parametric equations for the tangent line to the curve at the point P?

12. In this problem, $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ and $x'(t)$, $y'(t)$ and $z'(t)$ all exist.

(a) Suppose $\|\vec{r}(t)\|$ is a constant. Explain why $\vec{r}\,'(t)$ is perpendicular to $\vec{r}(t)$.

(b) Give an example of an $\vec{r}(t)$ which is not a constant but $\|\vec{r}(t)\|$ is a constant.

(c) Suppose $\|\vec{r}(t)\|$ is not a constant. Using an example, explain why it is not necessarily true that $\vec{r}\,'(t)$ is perpendicular to $\vec{r}(t)$.

Answers for §8

1. ... is *an ellipse* that is in a plane parallel to the *xy-plane*.

2. ... is *an ellipse* that lies in the plane $z = 3$.

3. $(1, 1, 1)$ and $(2, 4, 8)$

4. (a) $(0, 0, 0)$ and $(4/9, 2/3, 10/9)$

   (b) No

5. Yes, both particles are at $(3, 27, 7)$ when $t = 3$.

6. (a) π

   (b) $x = -3t,\ y = 4 + 4t,\ z = -5$

7. $x = t,\ y = 1,\ z = 2t$

8. $x = 1 - 2t,\ y = -3 + 2t,\ z = 2 - 3t$

9. $x = 2t,\ y = 2 - 3t,\ z = 1 + t$

10. $x = 1 + 2t,\ y = 1 + t,\ z = 1 + 3t$

11. $x = \sqrt{2} - \sqrt{2}\,t,\ y = \sqrt{2} + \sqrt{2}\,t,\ z = 8 - 8t$

12. (a) Since $\|\vec{r}(t)\|$ is a constant, $\|\vec{r}(t)\|^2$ is a constant. So there is some number c such that $x(t)^2 + y(t)^2 + z(t)^2 = c$ for all t. Taking derivatives of both sides of this equation and dividing by 2 gives

    $$x(t)x'(t) + y(t)y'(t) + z(t)z'(t) = 0.$$

    This is equivalent to $\langle x(t), y(t), z(t)\rangle \bullet \langle x'(t), y'(t), z'(t)\rangle = 0$. Hence, $\vec{r}(t) \bullet \vec{r}\,'(t) = 0$. Therefore, $\vec{r}\,'(t)$ is perpendicular to $\vec{r}(t)$.

    (b) One example is $\vec{r}(t) = \langle \cos t, \sin t, 1\rangle$.

    (c) An easy example is $\vec{r}(t) = \langle t, t, 1\rangle$ provided $t \neq 0$. Here, $\vec{r}\,'(t) = \langle 1, 1, 0\rangle$ so that $\vec{r}(t) \bullet \vec{r}\,'(t) = 2t \neq 0$ if $t \neq 0$. So $\vec{r}\,'(t)$ is not perpendicular to $\vec{r}(t)$ for $t \neq 0$. Another example is $\vec{r}(t) = \langle e^t, e^t, e^t\rangle$. In this second example, $\vec{r}(t) \bullet \vec{r}\,'(t) = 3e^{2t} \neq 0$ for every t, so $\vec{r}\,'(t)$ is not perpendicular to $\vec{r}(t)$ for every t.

---

§9. Homework Set 9: Arc length; Unit Tangent and Unit Normal; Velocity, Acceleration and Speed

FQ 1. Find the arc length of the graph of

$$\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle, \quad 0 \le t \le 2\pi.$$

FQ 2. Determine the arc length of the curve defined by the parametric equations

$$x = 3\cos(2t), \quad y = 4\cos(2t), \quad z = 5\sin(2t),$$

from $t = 0$ to $t = 2$.

FQ 3. Find the length of the curve given by

$$x = 2 + 3\sin t, \quad y = 1 - 4\sin t, \quad z = 7 + 5\cos t,$$

from $t = 0$ to $t = 20$.

FT 4. Find the arc length of the curve traced by

$$x = 7\,t - \cos t, \quad y = t + 7\cos t, \quad z = 5\sqrt{2}\,\sin t, \quad \text{where } 0 \le t \le \pi.$$

FQ 5. What is the length of the curve $x = 3\sin t$, $y = 4 - 4\sin t$ and $z = 5\cos t$ from $t = 0$ to $t = 2\pi$. Simplify your answer.

FT 6. Calculate the arc length of the curve defined by the parametric equations

$$x = 2 - 5\sin t - 3\cos t \quad \text{and} \quad y = 4 + 3\sin t - 5\cos t,$$

from $t = 0$ to $t = 2$.

FT 7. Find the arc length of the curve traced by

$$x = t^2 + t + 1, \quad y = t^2 + t - 2, \quad z = t^2 + t, \quad \text{where } 0 \le t \le 1.$$

FT 8. Find the arc length of the curve traced by

$$x = t^3 - 8\,t, \quad y = (4/3)\,t^3 + 6\,t, \quad z = 5\,t^2 - 3, \quad \text{where } 0 \le t \le 3.$$

FQ 9. Calculate the length of the curve given by

$$x = \frac{t^3}{3} - t, \quad y = -t^2 + 2, \quad z = \frac{t^3}{3} + t, \quad \text{where } 0 \le t \le 1.$$

FQ 10. Calculate the arc length of the curve given by

$$x(t) = \frac{2\,t^3}{3} - t^2, \quad y(t) = \frac{2\,t^3}{3} + t^2, \quad \text{and} \quad z(t) = \frac{t^3}{3} - t, \quad \text{where } 0 \le t \le 2.$$

FT 11. Determine the arc length of the curve defined by the parametric equations

$$x = 2t, \quad y = t^2, \quad z = \frac{t^3}{3},$$

from $t = 0$ to $t = 2$.

FT 12. Find the arc length of the curve traced by

$$x = 5\cos t, \quad y = 2\cos t + 6\sin t, \quad z = 4\cos t - 3\sin t, \quad \text{where } 0 \le t \le 2.$$

FT 13. Find the arc length of the curve traced by

$$x = 4\,e^t \sin t, \quad y = 4\,e^t \cos t, \quad \text{and} \quad z = 7\,e^t, \quad \text{where } 0 \le t \le 1.$$

FQ 14. Calculate the arc length of the curve given by

$$x = e^t + t, \;\; y = e^t - t, \;\; z = 4e^{t/2}, \quad \text{for } 0 \le t \le 1.$$

FQ 15. Calculate the arc length of the curve given by

$$x = e^{2t} + 2t, \;\; y = e^{2t} - 2t, \;\; z = 4e^t, \quad \text{for } 0 \le t \le 1.$$

FQ 16. Determine the arc length of the curve defined by the parametric equations

$$x = 2e^t - 10e^{-t}, \quad y = 11e^t - 5e^{-t}, \quad z = 10t + 3,$$

from $t = 0$ to $t = 1$.

FQ 17. Find the length of the curve given by

$$\overrightarrow{r}(t) = \langle 3t + 4\sin t, 4t - 3\sin t, 5\cos t\rangle$$

for $0 \le t \le \pi$.

FQ 18. (a) Calculate the length of the curve given by

$$x = 4t + 6\cos t, \quad y = 3t - 8\cos t, \quad z = 10\sin t$$

from $t = 0$ to $t = 2$.

(b) Find parametric equations for the tangent line to the curve given in part (a) at the point $(6, -8, 0)$.

FQ 19. (a) The parametric equations

$$x = -2\sin t, \quad y = 5t, \quad z = 2\cos t, \quad \text{for } 0 \le t \le \pi$$

describe a curve. Calculate the arc length of this curve.

(b) Find the parametric equations for the tangent line to the curve in part (a) at the point where $t = \pi/2$.

FQ 20. (a) Calculate the arc length of the curve given by

$$x = 3\sin t, \quad y = 5\cos t, \quad z = 4\sin t,$$

from $t = 0$ to $t = \pi$.

(b) Calculate the unit normal vector $\overrightarrow{N}(\pi/2)$ for the curve given in part (a) at $t = \pi/2$.

FT 21. A curve is described by the position vector $\vec{r}(t) = t^2\,\vec{i} - 5\,\vec{j} + \sqrt{t}\,\vec{k}$. Calculate a unit tangent vector for this curve at $t = 1$.

FQ 22. Calculate the unit tangent $\vec{T}(t)$ and unit normal $\vec{N}(t)$ for $\vec{r}(t) = \langle 3\cos t, 3\sin t, t\rangle$ at $t = 0$. Simplify your answers.

F 23. Calculate the unit tangent $\vec{T}(t)$ if $\vec{r}(t) = \langle te^t, t, te^t\rangle$.

FT 24. Let $\vec{r}(t) = \langle t^2, -t^3, 2t^2 + 2t - 1\rangle$. Then at $t = 1$, $\vec{r} = \langle 1, -1, 3\rangle$. Find a unit vector that is tangent to the curve given by $\vec{r}(t)$ at $t = 1$ (that is if its tail is placed on the curve at the point $(1, -1, 3)$)).

FT 25. If the velocity vector at time t of a particle is $\vec{v}(t) = (\sin t)\,\vec{i} + (\cos t)\,\vec{j} + \vec{k}$ and its initial position is at the origin (that is $\vec{r}(0) = (0, 0, 0)$), find its position at time $t = \pi$.

FT 26. Let $\vec{r}(t) = \langle 6t - 1, t^3, 3t^2\rangle$ be the position vector of a moving particle at time t.

(a) Calculate the velocity of the particle at time t.

(b) Calculate a unit vector that is tangent to the curve (the curve given by the position vector $\vec{r}(t)$) at time $t = 0$.

(c) Determine the length of the curve from $t = 0$ to $t = 1$.

FT 27. (a) Let $\vec{r}(t) = \langle t^2\sin(t), -t^2\cos(t), 2t\rangle$ be the position vector of a moving particle at time t. Calculate the velocity $\vec{v}(t)$ for the particle.

(b) Calculate the speed of the particle in part (a). Your answer should be in terms of t. Simplify your answer.

(c) Calculate the length of the curve traced by the moving particle from time $t = 0$ to $t = 3$.

FT 28. Let a particle's position vector at time t be given by $\vec{r}(t) = \langle \cos^2 t, \cos t\sin t, 2\sqrt{2}\,t\rangle$.

(a) Show that its speed is a constant.

(b) Calculate the length of the curve traced by $\vec{r}(t)$ from $t = 0$ to $t = 1$. (You should at least explain what you would do even if you could not do part (a).)

FT 29. A particle has velocity vector $\vec{v}(t) = \langle -4t^3, 2t - 4, 4\cos(\pi t)\rangle$.

(a) What is the particle's speed at $t = 1$? Simplify your answer.

(b) What is the particle's position at $t = 1$ if its initial position is given by $\vec{r}(0) = \langle 1, 2, 3\rangle$? You may use a point or a vector for your answer, but simplify your answer.

Answers for §9

1. 10π

2. 20
3. 100
4. 10π
5. 10π
6. $2\sqrt{34}$
7. $2\sqrt{3}$
8. 75
9. $4\sqrt{2}/3$
10. 10
11. $20/3$
12. $6\sqrt{5}$
13. $9e - 9$
14. $\sqrt{2}\,e$
15. $\sqrt{2}\,(e^2 + 1)$
16. $5\sqrt{5}\left(e - \dfrac{1}{e}\right)$
17. $5\sqrt{2}\,\pi$
18. (a) $10\sqrt{5}$
    (b) $x = 6 + 4t,\ y = -8 + 3t,\ z = 10t$
19. (a) $\sqrt{29}\,\pi$
    (b) $x = -2,\ y = (5\pi/2) + 5t,\ z = -2t$
20. (a) 5π
    (b) $\langle -3/5, 0, -4/5 \rangle$
21. $\langle 4/\sqrt{17}, 0, 1/\sqrt{17} \rangle$

22. $\overrightarrow{T}(t) = \dfrac{1}{\sqrt{10}}\langle 0,3,1\rangle$ and $\overrightarrow{N}(t) = \langle -1,0,0\rangle$

23. $\dfrac{1}{\sqrt{2t^2+4t+3}}\langle t+1,1,t+1\rangle$

24. $(1/7)\langle 2,-3,6\rangle$

25. $(2,0,\pi)$

26. (a) $\langle 6,3t^2,6t\rangle$
    (b) $\langle 1,0,0\rangle$
    (c) 7

27. (a) $\langle t^2\cos t + 2t\sin t, t^2\sin t - 2t\cos t, 2\rangle$
    (b) t^2+2
    (c) 15

28. (a) For this problem, $\overrightarrow{r}'(t) = \langle -2\sin t\cos t, \cos^2 t - \sin^2 t, 2\sqrt{2}\rangle$. So the speed is

$$\|\overrightarrow{r}'(t)\| = \left(\cos^4 t + \sin^4 t + 2\sin^2 t\cos^2 t + 8\right)^{1/2} = \left((\cos^2 t + \sin^2 t)^2 + 8\right)^{1/2} = \sqrt{9} = 3.$$

    (b) 3

29. (a) 6
    (b) $(0,-1,3)$

---

§10. Homework Set 10: Limits of Multivariable Functions

1. Describe the domain of the functions. (An appropriate picture will suffice.)
   (a) $f(x,y) = xy/(x^2+y^2)$
   (b) $f(x,y) = \sqrt{x^2-y^2}$
   (c) $f(x,y) = \sin(x-y)$
   (d) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$
   (e) $f(x,y,z) = \sqrt{4-(x^2+y^2+z^2)}$

FQ 2. Let
$$f(x,y,z) = \sqrt{23-x-3y+2z}.$$

Put a check mark in the box next to the best description for the domain of $f(x, y, z)$ from among the following.

- ☐ the points inside a circle of radius $\sqrt{23}$ centered at the origin
- ☐ the points to the right of a line
- ☐ the points inside a cone with vertex on the z-axis
- ☐ the points above a plane
- ☐ the points to the right of a parabola
- ☐ the points above a paraboloid with vertex on the z-axis

FQ 3. Graph the domain of $f(x, y) = \ln\left(1 - x^2 - y^2\right)$. Use solid lines for portions of the boundary included in the domain and dashed lines for portions not included. (You should use solid lines for the axes as well.)

4. Calculate

$$\lim_{(x,y)\to(1,1)} \frac{xy^3}{\sqrt{3x^2 - 4y^2}}.$$

FQ 5. Explain why the following limit does not exist. Use complete English sentences.

$$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2 + y^2}}$$

FQ 6. Explain why the limit below does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^4 + y^4}.$$

FQ 7. The limit below exists. Calculate its value. Be sure to show your work.

$$\lim_{(x,y)\to(0,0)} \frac{1 - (x^2 + y^2 - 1)^2}{4 - (x^2 + y^2 + 2)^2}$$

FQ 8. The following limit exists. Calculate the value of the limit.

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - 3x^2 + x^2 y - 3y^2}{x^2 + y^2}$$

FQ 9. Determine if the limit below exists. If it does, find its value. If not, explain why not.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{3x^2 + 2y^2}$$

FT 10. The following limit exists. What is it's value? Be sure to justify your answer.

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + \sin(x^2 + y^2)}{x^2 + y^2}.$$

FQ 11. The limit below exists. Calculate its value.

$$\lim_{(x,y)\to(0,0)} \frac{x\left(x+y\sqrt{x^2+y^2}\,\right)+y^2}{x^2+y^2}.$$

FQ 12. Calculate the value of

$$\lim_{(x,y)\to(0,0)} \frac{x^2(x+1)+y^2(y+1)}{x^2+y^2}.$$

FQ 13. Calculate the value of

$$\lim_{(x,y)\to(0,0)} \frac{\sin\left(2x^2+2y^2\right)}{3x^2+3y^2}.$$

FT 14. Justify that the limit below does not exist by finding two different paths to the origin that give two different limiting values. Indicate the limiting paths you are using and the limits you are getting along those paths.

$$\lim_{(x,y)\to(0,0)} \frac{3xy+2x^2y}{x^2+y^2-5x^2y^2}$$

FT 15. Justify that the following limit does NOT exist.

$$\lim_{(x,y)\to(0,0)} \frac{x+xy}{\sqrt{x^2+y^2}}$$

FT 16. Without using polar coordinates, explain why the following limit does NOT exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^5+y^5+4x^3y}{(x^2+2y^2)^2}.$$

FT 17. Explain why the following limit does NOT exist.

$$\lim_{(x,y)\to(0,0)} \frac{3x^4-2xy+3y^2}{x^4+y^2}.$$

Show appropriate work to back up any claims you make.

FT 18. Let

$$f(x,y) = \frac{x^3y}{x^4+y^4}.$$

Does $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exist? If so, what is it? If not, why not?

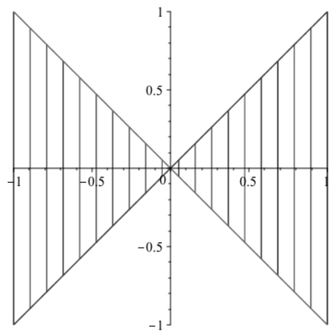
FQ 19. Justify below why the limit

$$\lim_{(x,y)\to(0,0)} f(x,y), \quad \text{where } f(x,y) = \frac{x^2y}{2x^4+3y^2},$$

does not exist by showing that as (x,y) approaches $(0,0)$ along two different paths, the function $f(x,y)$ approaches two different values. Indicate the paths you are considering (either in terms of x and y or in terms of a parameter t).

Answers for §10

1. (a) Every point on the xy-plane besides $(0,0,0)$

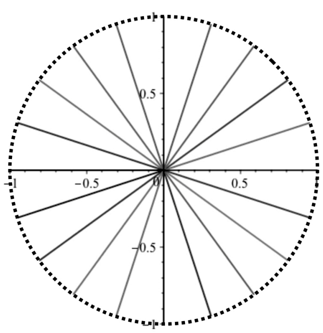
   (b) 

   (c) Every point in the xy-plane

   (d) Every point in space (the xyz-coordinate system)

   (e) Every point inside or on the sphere of radius 2 centered at the origin

2. the points above a plane

3. 

4. -1

5. Let $f(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$. For points approaching the origin along the positive x-axis, the value of $f(x,y)$ approaches 1. For points approaching the origin along the negative x-axis, the value of $f(x,y)$ approaches -1. Since these limiting values are different, the limit does not exist.

6. Let $f(x,y) = \dfrac{x^3 y}{x^4+y^4}$. For points approaching the origin along the line $y = x$, the value of $f(x,y)$ approaches $1/2$. For points approaching the origin along the line $y = -x$, the value of $f(x,y)$ approaches $-1/2$. Since these limiting values are different, the limit does not exist.

7. $-1/2$

8. -3

9. The limit does not exist. Let $f(x,y) = \dfrac{xy}{3x^2+2y^2}$. For points approaching the origin along the line $y = x$, the value of $f(x,y)$ approaches $1/5$. For points approaching the origin along the line $y = 0$, the value of $f(x,y)$ approaches 0. Since these limiting values are different, the limit does not exist.

10. 1

11. 1

12. 1

13. $2/3$

14. Let $f(x,y) = \dfrac{3xy + 2x^2y}{x^2+y^2-5x^2y^2}$. For points approaching the origin along the line $y = 0$, the value of $f(x,y)$ approaches 0. For points approaching the origin along the line $y = x$, the value of $f(x,y)$ approaches $3/2$. Since these limiting values are different, the limit does not exist.

15. Let $f(x,y) = \dfrac{x+xy}{\sqrt{x^2+y^2}}$. For points approaching the origin along the line $x = 0$, the value of $f(x,y)$ approaches 0. For points approaching the origin along the positive x-axis, the value of $f(x,y)$ approaches 1. Since these limiting values are different, the limit does not exist.

16. Let $f(x,y) = \dfrac{x^5+y^5+4x^3y}{(x^2+2y^2)^2}$. For points approaching the origin along the line $y = x$, the value of $f(x,y)$ approaches $4/9$. For points approaching the origin along the line $y = 0$, the value of $f(x,y)$ approaches 0. Since these limiting values are different, the limit does not exist.

17. Let $f(x,y) = \dfrac{3x^4-2xy+3y^2}{x^4+y^2}$. For points approaching the origin along the line $y = 0$, the value of $f(x,y)$ approaches 3. For points approaching the origin along the line $y = x$, the value of $f(x,y)$ approaches 1. Since these limiting values are different, the limit does not exist.

18. No, the limit does not exist. Let $f(x,y) = \dfrac{x^3y}{x^4+y^4}$. For points approaching the origin along the line $y = 0$, the value of $f(x,y)$ approaches 0. For points approaching the origin along the line $y = x$, the value of $f(x,y)$ approaches $1/2$. Since these limiting values are different, the limit does not exist.

19. Let $f(x,y) = \dfrac{x^2y}{2x^4+3y^2}$. For points approaching the origin along the line $y=0$, the value of $f(x,y)$ approaches 0. For points approaching the origin along the path described by $x=t$ and $y=t^2$, the value of $f(x,y)$ approaches $1/5$. Since these limiting values are different, the limit does not exist.

---

§11. Homework Set 11: Partial Derivatives

FQ 1. Calculate $f_{xyx}(-1,10)$ where $f(x,y) = x^3y - 3xy^2$.

FQ 2. Calculate $f_{xy}(-1,2)$ if $f(x,y) = x^2y - y^2$.

FT 3. Calculate $\partial f/\partial y$ where $f(x,y) = x^2\sin(xy)$.

FQ 4. Let $f(x,y) = ye^{xy}$. Calculate both $f_{xx}(x,y)$ and $f_{xx}(0,1)$.

FQ 5. (a) Calculate $\displaystyle\lim_{h\to 0}\frac{e^{x(y+h)} - e^{xy}}{h}$.

(b) Did your answer to part (a) have any connection to partial derivatives? If not, determine how part (a) is related to partial derivatives and redo the problem.

FF 6. (a) Let

$$f(x,y) = \lim_{h\to 0}\frac{(x+2y+h)^{3/2} - (x+2y)^{3/2}}{h}.$$

Calculate $f(2,1)$.

(b) Did your answer to part (a) have any connection to partial derivatives? If not, determine how part (a) is related to partial derivatives and redo the problem.

FT 7. Calculate f_{xxxxyy} if $f(x,y) = x^3y^2\sqrt{y}\,\sin(y)e^y\ln y$.

FT 8. Calculate f_{xxyy} for $f(x,y) = x^3y^2 + xe^{\cos(y^2+y)}\sqrt{\sin y} - (x+y)^4$.

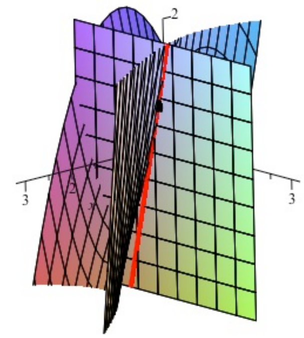
FQ 9. Let $f(x,y) = x^3y + \cos(x^2 + \ln x)$. Calculate $\dfrac{\partial^3 f}{\partial^2 x\,\partial y}$. Simplify your answer.

FT 10. Calculate $\dfrac{\partial^5 f(x,y)}{\partial^2 x\,\partial^3 y}$ where $f(x,y) = xy^2\sin(x^2) + 4y^3 + \sqrt{x}$.

FQ 11. Calculate $f_{yx}(2,-1)$ if $f(x,y) = x^3y^2 + 8y - \ln\left(x - \sqrt{x^2+\sqrt{x}}\right)$.

FQ 12. If $f(x,y) = \dfrac{x}{y}$, then what is the value of $\dfrac{\partial^2}{\partial y\,\partial x}f(x,y)$ and the value of $f_{xy}(1,2)$?

FQ 13. The point $A = (1, 1, 1)$ is on the hyperbolic paraboloid $z = 2y^2 - x^2$. The plane $x = 1$ intersects the hyperbolic paraboloid $z = 2y^2 - x^2$ in a curve that passes through A. What is the slope of the tangent line to this curve at the point A? Show work. (View the plane from the side shown in the picture.)



FQ 14. A point moves along the intersection of the elliptic paraboloid $z = x^2 + 3y^2$ and the plane $y = 1$. At what rate is z changing with respect to x when the point is at $(2, 1, 7)$?

FQ 15. A point moves along the intersection of the surface $z = x/(x - y)$ and the plane $x = 2$. At what rate is z changing with respect to y when the point is at $(2, 1, 2)$?

FT 16. The plane $x = 1$ intersects the surface $z = 3x^2 + 2\sqrt{y}$ in a curve. What is the slope of the tangent line to that curve at the point $(x, y, z) = (1, 1, 5)$?

Selected Answers for §11

1. -60

2. 4

3. $x^3 \cos(xy)$

4. $f_{xx}(x, y) = y^3 e^{xy}$ and $f_{xx}(0, 1) = 1$

5. (a) xe^{xy}

   (b) The limit in part (a) is the definition of the partial derivative of e^{xy} with respect to y.

6. (a) 3

   (b) The value of $f(2, 1)$ is the partial derivative of $(x + 2y)^{3/2}$ with respect to x evaluated at the point $(2, 1)$. (The value of $f(2, 1)$ could also be viewed as the derivative of the single variable function $x^{3/2}$ evaluated at $x = 4$. Why?)

7. 0

8. $12x + 24$

9. $6x$

10. 0

11. -24

12. $\dfrac{\partial^2}{\partial y\,\partial x} f(x,y) = -\dfrac{1}{y^2}$ and $f_{xy}(1,2) = -\dfrac{1}{4}$

13. 4

    Solution 1. The curve is on the plane $x = 1$, so x is a constant. The variable y is changing, and z can be viewed as just a function of y. The problem then is asking for the value of $\partial z/\partial y$ at the point $(1,1,1)$ (where $y = 1$). Since $\partial z/\partial y = 4y$, we see that the slope at $(1,1,1)$ is 4. Note that the positive y-axis is pointing to the right when viewing the plane $x = 1$ as in the picture, so the slope should be positive as it appears in the picture. ■

    Solution 2. The plane $x = 1$ intersects the surface $z = 2y^2 - x^2$ in the curve $z = 2y^2 - 1$. In other words, z is just a single variable function of the variable y when $x = 1$. So the slope is the derivative of z with respect to y at the point we are interested in (where $y = 1$). Given $z = 2y^2 - 1$, we get $dz/dy = 4y = 4$ at $y = 1$. ■

14. 4

    Solution. Since y is a constant, namely $y = 1$, on the plane $y = 1$, the value of z is changing as x changes on this plane. Since we are considering points on the elliptic paraboloid $z = x^2 + 3y^2$, the problem is asking for the value of $\partial z/\partial x$ at the point $(2,1,7)$. Note that the wording asking for a rate of change in z with respect to x is a standard way of asking for a derivative of z with respect to the variable x or, in other words, $\partial z/\partial x$. Since $\partial z/\partial x = 2x = 4$ at $(2,1,7)$ (where $x = 2$), the answer is 4. ■

15. 2

16. 1

---

§12. Homework Set 12: Directional Derivatives

FT 1. Calculate $\nabla f(2,1)$ where $f(x,y) = x^2 + y^3$.

FT 2. Find the directional derivative of $f(x,y) = x^2y + x + 2$ at the point $P = (1,1)$ in the direction of $\overrightarrow{v} = -i + j$.

FT 3. Calculate the directional derivative of $f(x,y) = x^5 - 2y^5$ at the point $(x,y) = (1,1)$ in the direction $\overrightarrow{v} = \langle -3, 4\rangle$. Simplify your answer.

FQ 4. Calculate the directional derivative of $f(x,y) = x^2y - 3y$ at $P = (1,1)$ in the direction of $Q = (4,5)$.

FQ 5. Find the directional derivative of $f(x,y) = \sqrt{xy}$ at $P = (2,8)$ in the direction of $Q = (5,4)$.

FF 6. Calculate the directional derivative of $f(x,y) = xe^y + y^2$ at the point $(-1,0)$ in the direction $\langle 1,-1\rangle$.

FT 7. Let $\overrightarrow{v} = \langle 1,3\rangle$ and $f(x,y) = x^2y + y^2$. Calculate the directional derivative of $f(x,y)$ in the direction $\overrightarrow{v}$ at the point $(1,-1)$.

FQ 8. Find the directional derivative of $f(x,y) = xy + (1/x)$ at the point $P = (1,1)$ in the direction of the point $Q = (2,2)$.

FT 9. Calculate the directional derivative of $f(x,y) = (\sin x)(\cos y)$ at the point $(\pi,0)$ in the direction of the vector $\overrightarrow{v} = \langle 8,-6\rangle$. Simplify your answer.

FQ 10. Find the directional derivative of $f(x,y,z) = xy + yz^2$ at $(-4,1,2)$ in the direction of the vector $\langle 3,-6,2\rangle$. Simplify your answer.

FF 11. (a) Let $f(x,y,z) = x^2 + 2y^2z + y^2 - 2$. Note that the point $P = (1,1,0)$ is a point on the graph of $f(x,y,z) = 0$. Calculate the directional derivative of $f(x,y,z)$ at the point P in the direction of the vector $\langle 6,-3,-2\rangle$. Simplify your answer.

(b) Calculuate the maximum value of the directional derivative for $f(x,y,z) = x^2 + 2y^2z + y^2 - 2$ at the point $P = (1,1,0)$?

FT 12. For both parts of this problem, $f(x,y) = y\sin(xy)$. Simplify your answers.

(a) Find the directional derivative of $f(x,y)$ at the point $(0,1)$ in the direction of $\langle -2,1\rangle$.

(b) There are infinitely many different values for the directional derivative of $f(x,y)$ at the point $(0,1)$ (since there are infinitely many directions that can be used to compute the directional derivative). Which of these is maximal? In other words, what is the largest value of the directional derivative of $f(x,y)$ at the point $(0,1)$?

FT 13. For both parts of this problem, $f(x,y) = x^2 - y^2 + 1$.

(a) Find the directional derivative of $f(x,y)$ at the point $(0,1)$ in the direction of $\langle 1,1\rangle$.

(b) There are infinitely many different values for the directional derivative of $f(x,y)$ at the point $(0,1)$ (since there are infinitely many directions that can be used to compute the directional derivative). Which of these is maximal? In other words, what is the largest value the directional derivative of $f(x,y)$ at the point $(0,1)$?

FT 14. For both parts of this problem, $f(x,y) = x^2y^2 + x + y$.

(a) Find the directional derivative of $f(x,y)$ at the point $(2,1)$ in the direction of $\langle 0,3\rangle$.

(b) There are infinitely many different values for the directional derivative of $f(x,y)$ at the point $(2,1)$ (since there are infinitely many directions that can be used to compute the directional derivative). Which of these is maximal? In other words, what is the largest value of the directional derivative of $f(x,y)$ at the point $(2,1)$?

FQ 15. (a) Calculate the directional derivative of $f(x,y) = x - xy$ at the point $(1,2)$ in the direction of $\langle 2,1\rangle$.

(b) There are infinitely many different values for the directional derivative of $f(x,y) = x - xy$ at the point $(1,2)$ (since there are infinitely many directions that can be used to compute the directional derivative). Which of these is maximal? In other words, what is the largest value of the directional derivative of $f(x,y) = x - xy$ at the point $(1,2)$?

FQ 16. In what direction $\overrightarrow{u}$ does $f(x,y) = 1 - x^2 - 3y^2$ *decrease* most rapidly at $P = (2,-1)$? Express your answer as a unit vector $\overrightarrow{u}$.

FQ 17. Find a unit vector $\overrightarrow{u} = \langle a,b\rangle$ in the direction in which $f(x,y) = x^3y^4$ increases most rapidly at the point $(-1,1)$ and find the rate of change of $f(x,y)$ at $(-1,1)$ in that direction.

FT 18. The plane $y = x$ passes through the point $(1,1,2)$ and cuts (intersects) the surface of $z = x^4 - 2xy + 3y^3$ in a curve. A tangent line to this curve is drawn at the point $(1,1,2)$ in the plane $y = x$. What is the slope of this tangent line?

FT 19. The point $A = (4,1,8)$ is on the surface given by the graph of $z = x^2y - 2xy^3$ and is directly above the point $B = (4,1,0)$ in the xy-plane. A plane $\mathcal{P}$ passes through these two points and the point $C = (6,1,0)$ in the xy-plane (so $\mathcal{P}$ is the plane passing through the points A, B and C). This plane $\mathcal{P}$ cuts the surface in a curve that passes through the point $A = (4,1,8)$. A line ℓ is tangent to the curve at the point A. What is the slope of ℓ? Justify your answer with correct work. (Note: there are two answers depending on the side of the plane being viewed; either answer is fine.)

FQ 20. The point $A = (2,1,2)$ is on the surface given by the graph of $z = x^2 - y^2x$ and is directly above the point $B = (2,1,0)$ in the xy-plane. A plane $\mathcal{P}$ passes through these two points and the point $C = (5,-3,0)$ in the xy-plane (so $\mathcal{P}$ is the plane passing through the points A, B and C). This plane $\mathcal{P}$ cuts the surface in a curve that passes through the point $A = (2,1,2)$. A line ℓ is tangent to the curve at the point A. What is the slope of ℓ? Justify your answer with correct work. (Note: there are two answers depending on the side of the plane being viewed; either answer is fine.)

FT 21. The point $A = (2,1,6)$ is on the surface given by the graph of $z = x + 4\,x^2y - 6\,x\,y^2$ and is directly above the point $B = (2,1,0)$ in the xy-plane. A plane $\mathcal{P}$ passes through these two points and the point $C = (5,-3,0)$ in the xy-plane (so $\mathcal{P}$ is the plane passing through the points A, B and C). This plane $\mathcal{P}$ cuts the surface in a curve that passes through the point $A = (2,1,6)$. A line ℓ is tangent to the curve at the point A. What is the slope of ℓ? Simplify your answer and justify your answer with correct work. (Note: there are two answers depending on the side of the plane being viewed; either answer is fine.)

FQ 22. The point $A = (1,2,-6)$ is on the surface given by the graph of $z = x^2y - y^3$ and is directly above the point $B = (1,2,0)$ in the xy-plane. A plane $\mathcal{P}$ passes through these two points and the point $C = (4,-2,0)$ in the xy-plane (so $\mathcal{P}$ is the plane passing through the points A, B and C). This plane $\mathcal{P}$ cuts the surface in a curve that passes through the point

$A = (1, 2, -6)$. A line ℓ is tangent to the curve at the point A. What is the slope of ℓ? Justify your answer with correct work. (Note: there are two answers depending on the side of the plane being viewed; either answer is fine.)

FT 23. Determine a *unit* vector $\overrightarrow{u}$ for which the directional derivative $D_{\overrightarrow{u}}f(1, 1)$ is maximal where

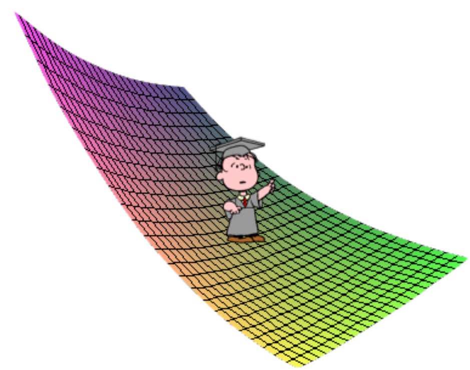
$$f(x, y) = x^3 + 2xy + 2y^5.$$

FT 24. Determine a *unit* vector $\overrightarrow{u}$ for which the directional derivative $D_{\overrightarrow{u}}f(1, 0)$ is *minimal* where

$$f(x, y) = 2x^2 + 3xy - 4y^3.$$

FF 25. A drop of water is placed gently onto the surface of $z = 3y^2 - 4x^2$ at the point $(1, 1, -1)$. In what direction does the drop begin to move (assume it goes downward in the direction of the steepest descent)? Express your answer as a unit vector $\langle a, b\rangle$ (so the raindrop will go in this direction along the surface).

FT 26. A Calculus student stands on the surface $z = (x - 2y)^3$ at the point $(3, 1, 1)$ and oils his feet to be silly. In what direction does the shadow of his feet slide in the xy-plane? This question is asking for the direction (answer with a vector with two components) in which the directional derivative of $(x - 2y)^3$ is minimal at the point $(3, 1, 1)$. You do *not* need to put your answer in the form of a unit vector.



FT 27. There are infinitely many different values for the directional derivative of $f(x, y) = x^2y - 2xy^3$ at the point $(4, 1)$ (since there are infinitely many directions that can be used to compute the directional derivative). Which of these is maximal? In other words, what is the largest value of the directional derivative of $f(x, y)$ at the point $(4, 1)$?

FT 28. There are infinitely many different values for the directional derivative of $f(x, y) = x^2 - y^3x$ at the point $(x, y) = (4, -1)$ (since there are infinitely many directions that can be used to compute the directional derivative). Which of these is maximal? In other words, what is the largest value of the directional derivative of $f(x, y)$ at the point $(4, -1)$?

FF 29. The plane $x + 2y = 3$ intersects the surface $z = x^5 + y^3$ in a curve. The point $(1, 1, 2)$ is on the plane and the surface, so $(1, 1, 2)$ is on the curve. Find parametric equations for the tangent line to the curve at the point $(1, 1, 2)$.

FF 30. The two surfaces $z = x^2 - y^2 + 8$ and $z = 3x^2 + y^2$ intersect in a curve. The point $P = (\sqrt{2}, \sqrt{2}, 8)$ is on the curve. What are the parametric equations for the tangent line to the curve at the point P? (Hint: Find a parameterization for the curve.)

Selected Answers for §12

1. $\langle 4, 3\rangle$
2. $-\sqrt{2}$
3. -11
4. $-2/5$
5. $2/5$
6. $\sqrt{2}$
7. $-5/\sqrt{10}$ (or equivalently $-\sqrt{5/2}$)
8. $1/\sqrt{2}$
9. $-4/5$
10. $11/7$
11. (a) $2/7$
    (b) $2\sqrt{3}$
12. (a) $-2/\sqrt{5}$
    (b) 1
13. (a) $-\sqrt{2}$
    (b) 2
14. (a) 9
    (b) $\sqrt{106}$
15. (a) $-3/\sqrt{5}$
    (b) $\sqrt{2}$
16. $\dfrac{1}{\sqrt{13}}\langle 2, -3\rangle$

17. $\vec{u} = \langle 3/5, -4/5 \rangle$; rate of change is 5

18. $9/\sqrt{2}$

    Solution. The plane $y = x$ intersects the xy-plane in the line $y = x$ (in that plane). We can view the plane $y = x$ as passing through the vector $\langle 1, 1 \rangle$ in the xy-plane with its initial point at $(1, 1)$ in that plane (equivalently, the vector $\langle 1, 1, 0 \rangle$ in the xyz-coordinate system with initial point at $(1, 1, 0)$). The point $(1, 1, 2)$ is on the surface $z = x^4 - 2xy + 3y^3$ and directly above the point $(1, 1)$ in the xy-plane. The problem then is asking for the directional derivative of the function $f(x, y) = x^4 - 2xy + 3y^3$ at the point $(1, 1)$ in the direction of the vector $\vec{v} = \langle 1, 1 \rangle$. A unit vector $\vec{u}$ in the direction of $\vec{v}$ is $\vec{v}/\|\vec{v}\|$. Thus, we want $\vec{u} = (1/\sqrt{2})\langle 1, 1 \rangle$. Since $\nabla f = \langle 4x^3 - 2y, -2x + 9y^2 \rangle$, we have $\nabla f(1, 1) = \langle 2, 7 \rangle$. Hence, the answer is

    $$D_{\vec{u}} f(1, 1) = \langle 2, 7 \rangle \bullet \vec{u} = \frac{9}{\sqrt{2}}.$$

    Observe that one could take $\vec{v} = -\langle 1, 1 \rangle$ in this problem (i.e., the wording allows for this), which leads to an alternative answer of $-9/\sqrt{2}$. ■

19. 6 or -6

    Solution. The plane $\mathcal{P}$ passes through B and C in the xy-plane (where $z = 0$). Thus, $\mathcal{P}$ passes through the vector $\overrightarrow{BC} = \langle 2, 0, 0 \rangle$ with initial point at B. Since B is directly below the point $A = (4, 1, 8)$, which is on the plane $\mathcal{P}$ and on the surface $z = x^2y - 2xy^3$, the problem is asking for the directional derivative of $f(x, y) = x^2y - 2xy^3$ at the point A (or at the point $(x, y) = (4, 1)$ in the xy-plane) in the direction of the vector $\overrightarrow{BC}$ (or $\vec{v} = \langle 2, 0 \rangle$ in the xy-plane). Here, $\nabla f = \langle 2xy - 2y^3, x^2 - 6xy^2 \rangle$, so $\nabla f(4, 1) = \langle 6, -8 \rangle$. The unit vector in the direction of $\vec{v}$ is $\vec{u} = \langle 1, 0 \rangle$. Hence, the answer is

    $$D_{\vec{u}} f(1, 1) = \langle 6, -8 \rangle \bullet \vec{u} = 6.$$

    If one views the slope from one side of the plane $\mathcal{P}$, the slope will be 6. From the other side, it will be -6. (Note that this could be viewed as a partial derivative question since A, B and C are all on the plane $y = 1$. In other words, $\mathcal{P}$ is the plane $y = 1$, so this problem is just asking for $\partial z/\partial x$ when $(x, y) = (4, 1)$.) ■

20. ± 5

21. ± 13

22. $\pm 56/5$

23. $\dfrac{1}{13}\langle 5, 12 \rangle$

24. $-\dfrac{1}{5}\langle 4, 3 \rangle$

25. $\frac{1}{5}\langle 4,-3\rangle$

26. $\langle -1,2\rangle$ (or some positive number times this)

27. 10

28. 15

29. $x=1-2t,\ \ y=1+t,\ \ z=2-7t$

30. $x=\sqrt{2}-\sqrt{2}\,t,\ \ y=\sqrt{2}+\sqrt{2}\,t,\ \ z=8-8t$

---

§13. Homework Set 13: The Chain Rule

FT 1. Let $z=x^3y+y^2x-x$ where $x=e^{st}$ and $y=te^{st}$. Using the Chain Rule, calculate $\partial z/\partial s$ in terms of s and t (but you do not need to simplify your answer).

FQ 2. Let

$$z=x^2y-y^4,\quad x=s^2t^3-3st+\sin(s^2-1)\quad\text{and}\quad y=3t^3-2s+\cos(5t).$$

Calculate $\dfrac{\partial z}{\partial s}$ at the point where $s=1$ and $t=0$. Simplify your answer.

FT 3. Use the chain rule to calculate $\partial w/\partial t$ where

$$w=x^2y+y^2-x\cos(z)+\log(1+\sqrt{z}),$$

$$x=t^2-st+s\sin(s),\quad y=(s+t)^2,\quad z=s^3+2^{s+1}.$$

You do not need to express your answer in terms of t and s.

FT 4. Use the chain rule to calculate $\partial w/\partial t$ where

$$w=xy^2+y^3-xz\sin(z),$$

$$x=t^2-s^2t+\sqrt{s\sin s},\quad y=s^3-t^2,\quad z=s^3+2^{s+1}.$$

You do not need to express your answer in terms of t and s.

FT 5. Using the Chain Rule, compute $\dfrac{\partial z}{\partial t}$ where

$$z=xy^2+x^2+3y+5,\qquad x=s^2t+(s+3)^2e^{2s+1},$$

and

$$y=s(s-1)^4+\cos(4s+1)+t^2.$$

You do not need to put your answer in terms of s and t (the variables x and y can appear in your answer).

FQ 6. Calculate $\partial z/\partial\theta$ at the point $(\theta,\phi)=(0,0)$ given that $z=x^4+3xy-y^3+2$, where

$$x=3\theta+\cos\theta-(\sin\phi)\,e^{1/\sqrt{1+\phi}} \quad \text{and} \quad y=-\phi+(\theta+1)\cos\theta+(\sin\phi)\,e^{1/\sqrt{1+\phi}}.$$

Simplify your answer.

FT 7. Using the Chain Rule, compute $\dfrac{\partial w}{\partial t}$ where

$$w=x^2+xyz+x+2z, \qquad x=t\sin(\sqrt{s}\,)+2^s-t^2$$

$$y=2s+s^2\sin(t), \qquad \text{and} \qquad z=t^2s^3-2t$$

You do not need to put your answer in terms of s and t (the variables x, y, and z can appear in your answer).

FT 8. Use the chain rule to calculate $\partial w/\partial t$ where

$$w=x^2y-xz,$$

$$x=u^2v, \quad y=u\cos^2 u, \quad z=u-v$$

$$u=s^2-2s, \quad \text{and} \quad v=st.$$

You do *not* need to express your answer in terms of s and t.

FT 9. Use the chain rule to calculate $\partial w/\partial u$ where

$$w=xy^2z^3, \quad x=uv-\sin v, \quad y=u^2-v^2, \quad z=u^3+2u.$$

You do not need to express your answer in terms of u and v.

FT 10. Calculate $\dfrac{dw}{dt}$ at $t=0$, where

$$w=x^2y+xy^2+3\,z^2, \quad x=t^2+3\,t-1, \quad y=t^2+5\,t+2, \quad \text{and} \quad z=t^2\sin(t)-t.$$

Note that your answer should be a number.

FQ 11. Using the chain rule, calculate $\partial w/\partial u$ at $u=1$ and $v=-1$ given that

$$w=(x^2+y^2+z^2)^{3/2}, \quad x=3u-v^2, \quad y=6\big(\sqrt{\ln(|2v|)}-\sqrt{\ln 2}+1\big), \quad z=3u^2v^3.$$

Simplify your answer.

FQ 12. Suppose that $z=f(x,y)$ is differentiable at the point $(4,8)$ with $f_x(4,8)=1$ and with $f_y(4,8)=-2$. If $x=t^2$ and $y=t^3$, find dz/dt when $t=2$.

FT 13. Calculate $\partial w/\partial t$ at the point $(s,t)=(1,2)$ given that $w=x^2\,y-\cos(xy)\,z^2$ where

$$x=s+4\,t,\quad y=2\,s^2-t,\quad \text{and}\quad z=s\,t^2-4\,s.$$

FT 14. Calculate $\partial z/\partial t$ given that $z=y^2\sqrt{y}\sin(x+y)$, $x=3u+2v$, $y=v^3-12v^2+5v+16$, $u=r+2s$, $v=r^2+3r+5$, $r=w^3+\cos(w)$, and $s=3tw$. Use any method you want. You do not need to write your answer in terms of w and t (i.e., you may have other variables in your answer). Do NOT simplify.

FQ 15. Using the Chain Rule for functions in several variables, calculate $\dfrac{dw}{dt}$ when $t=1$ given

$$w=xy+xz-y^2,\quad x=t^2+1,\quad y=2t-3,\quad z=3t-2.$$

FQ 16. The height, width and length of a box are changing with time. The height of the box is increasing at a constant rate of 1 inch per second, the width is increasing at a constant rate of 2 inches per second, and the length is decreasing at a constant rate of 3 inches per second. At a particular moment, the box has height 2 inches, width 3 inches and length 4 inches. At what rate is the volume of the box changing at this same moment?

Answers for §13

1. $(3te^{3st}+t^2e^{2st}-1)te^{st}+(e^{3st}+2te^{2st})t^2e^{st}$

2. -8

3. $(2xy-\cos z)(2t-s)+2(x^2+2y)(s+t)$

4. $(y^2-z\sin z)(2t-s^2)+(2xy+3y^2)(-2t)$

5. $(y^2+2x)s^2+(2xy+3)(2t)$

6. 21

7. $(2x+yz+1)(\sin(\sqrt{s}\,)-2t)+xzs^2\cos(t)+(xy+2)(2ts^3-2)$

8. $(2xy-z)u^2s+xs$

9. $y^2z^3v+4xyz^3u+3xy^2z^2(3u^2+2)$

10. -15

11. 504

12. -20

13. -81

14. $18wy^2\sqrt{y}\cos(x+y)$

15. 14

16. 46 in^3/sec

---

§14. Homework Set 14: Tangent Planes

FT 1. Find an equation for the tangent plane to the hyperboloid of one sheet $x^2 - y^2 + 2z^2 = 1$ at the point $(3, 4, 2)$.

FT 2. Find an equation for the tangent plane to the hyperboloid of two sheets $x^2 - 2y^2 - z^2 = 1$ at the point $(2, -1, 1)$.

FT 3. Find an equation for the tangent plane to the hyperbolic paraboloid $z = y^2 - x^2$ at the point $(2, 1, -3)$.

FQ 4. Find an equation for the tangent plane to the cone $z^2 = x^2 - y^2$ at the point $(5, 3, 4)$.

FT 5. Find every point $P = (a, b, c)$ on the surface $z = (x + y)^3 x + x^2 - x$ such that the tangent plane to the surface at P is horizontal (i.e., the tangent plane is parallel to the $xy-$plane).

FT 6. Find an equation for the tangent plane to the surface $x^2 + xy - y^2 - z^2 = 4$ at the point $(2, 1, 1)$.

FT 7. Find an equation for the tangent plane to the surface $x^2 - 2y^2 = xyz^2$ at the point $(1, -1, -1)$.

FT 8. Find an equation for the tangent plane to the surface $x^3 - x\sin(y) + z^2 = 0$ at the point $(-1, 0, 1)$.

FT 9. Find an equation for the tangent plane to the surface $z = x^2y + xy^2 - 4$ at the point $(1, 2, 2)$.

FT 10. Find an equation for the tangent plane to the surface $x^4 + xy^2 - y^2z = 1$ at the point $(1, 1, 1)$.

FT 11. Find an equation for the tangent plane to the surface $z^2 = x^3 + y^2$ at the point $(2, 1, -3)$.

FQ 12. Calculate an equation for the tangent plane to $x^3 - y^2 + z^4 = 1$ at the point $(1, 1, 1)$.

FQ 13. Calculate an equation for the tangent plane to the hyperbolic paraboloid $z = 3y^2 - 2x^2$ at the point $(-1,-1,1)$.

FQ 14. Calculate an equation for the tangent plane to the surface

$$2(x-2)^2+(y-1)^2+(z-3)^2=10$$

at the point $(3,3,5)$.

FT 15. Find an equation for the tangent plane to the surface $x^2+3xy-2y^2+z^2=0$ at the point $(1,-1,2)$.

FT 16. Find each point P on the surface of the hyperboloid of one sheet given by $x^2-y^2+z^2=1$ for which the tangent plane to the surface at P is parallel to the plane $5x+y+5z=4$. There are two such points.

FT 17. There are two points on the ellipsoid $x^2+2y^2+3z^2=24$ where the tangent plane is parallel to the plane $2x+4y+6z=15$. What are those two points? Be sure to show work.

FT 18. The plane $\mathcal{P}$ given by $12\,x-11\,y-z=6$ is tangent to the surface $x^3=y^4+yz$ at a point $A=(2,y_0,z_0)$. Determine the values of y_0 and z_0. Your work should justify that the plane $\mathcal{P}$ is tangent to the surface at A.

FT 19. A plane is tangent to the ellipsoid

$$\frac{x^2}{12}+\frac{y^2}{18}+\frac{z^2}{6}=1$$

at the point $(2,3,-1)$. Find parametric equations for the *line* which is perpendicular to the tangent plane at $(2,3,-1)$.

FF 20. There are two points P and Q on the graph of $x^2-y^2=z^2-1$ such that the tangent plane at P and the tangent plane at Q are both parallel to the xz-plane. Determine the points P and Q.

FQ 21. Find parametric equations for the normal line to the ellipsoid $x^2+4y^2+4z^2=36$ at the point $(4,-1,2)$.

FT 22. Recall that the normal line to a surface at a point on the surface is perpendicular to the tangent plane to the surface at that point. Calculate the parametric equations for the normal line to the surface $z=2xy^2-3xy$ at the point $(1,1,-1)$.

FF 23. A parabola has some interesting properties, but we won't confuse you with them here. The purpose of this problem, though, is to show you that a paraboloid (at least the one given below) has similar properties.

(a) Let $P=(x_0,y_0,z_0)$ be a point on the paraboloid $4z=x^2+y^2$. Show that the distance from P to the point $(0,0,1)$ is the same as the distance from P to the plane $z=-1$. (Note that you should be showing that this is true for *every* point P on the paraboloid $4z=x^2+y^2$. In other words, simply use $P=(x_0,y_0,z_0)$ where $4z_0=x_0^2+y_0^2$.)

(b) Write down a vector that is perpendicular to the tangent plane to $4z = x^2 + y^2$ at $P = (x_0, y_0, z_0)$.

(c) Let $\overrightarrow{n}$ be the vector in part (b), let $\overrightarrow{u}$ be the vector from $P = (x_0, y_0, z_0)$ to $(0, 0, 1)$, and let $\overrightarrow{v} = \langle 0, 0, 1\rangle$. Explain why these 3 vectors are all parallel to or on the plane $y_0x - x_0y = 0$. (In other words, a line going in the direction of any one of these vectors will be on the plane $y_0x - x_0y = 0$ or the line will be parallel to the plane $y_0x - x_0y = 0$.)

(d) Given the vectors in part (c), explain why the smallest angle between vectors $\overrightarrow{n}$ and $\overrightarrow{u}$ is the same as the smallest angle between vectors $\overrightarrow{n}$ and $\overrightarrow{v}$.

Selected Answers for §14

1. $3x - 4y + 4z = 1$
2. $2x + 2y - z = 1$
3. $4x - 2y + z = 3$
4. $5x - 3y - 4z = 0$

   Comment: For fun, show that the tangent plane at every point (a, b, c) on the cone $z^2 = x^2 - y^2$ passes through the origin. This may be clear from a picture, but do this algebraically by computing the equation for the tangent plane at (a, b, c) and verifying that $(0, 0, 0)$ is a point on it.
5. $(0, 1, 0)$ and $(1/2, -1/2, -1/4)$
6. $5x - 2z = 8$
7. $3x + 3y - 2z = 2$
8. $3x + y + 2z = -1$
9. $8x + 5y - z = 16$
10. $5x - z = 4$
11. $6x + y + 3z = 4$
12. $3x - 2y + 4z = 5$
13. $4x - 6y - z = 1$

14. $x+y+z=11$

15. $x-7y-4z=0$

16. $(5/7,-1/7,5/7)$ and $(-5/7,1/7,-5/7)$

17. $(2,2,2)$ and $(-2,-2,-2)$

    Solution. Let $F(x,y,z)=x^2+2y^2+3z^2$, and let $P=(x_0,y_0,z_0)$ be one of the two points in the problem. The vector $\nabla F(x_0,y_0,z_0)=\langle 2x_0,4y_0,6z_0\rangle$ is in the direction of the normal to the tangent plane at P. Since this tangent plane is parallel to $2x+4y+6z=15$, which has a normal vector $\langle 2,4,6\rangle$, we must have

    $$\langle 2x_0,4y_0,6z_0\rangle = k\langle 2,4,6\rangle$$

    for some number $k\neq 0$. This gives $2x_0=2k$, $4y_0=4k$ and $6z_0=6k$. Therefore, $x_0=y_0=z_0=k$. In other words, $P=(k,k,k)$. On the other hand, P is on the ellipsoid $x^2+2y^2+3z^2=24$. Therefore, $k^2+2k^2+3k^2=24$ which leads to $k^2=4$ and, hence, $k=\pm 2$. We deduce then that P equals $(2,2,2)$ or $(-2,-2,-2)$. ■

18. $y_0=1$ and $z_0=7$

    Solution. Let $F(x,y,z)=x^3-y^4-yz$. Observe that $\nabla F=\langle 3x^2,-4y^3-z,-y\rangle$. Hence, a vector normal to the tangent plane at $A=(2,y_0,z_0)$ is $\nabla F(2,y_0,z_0)=\langle 12,-4y_0^3-z_0,-y_0\rangle$. Since the tangent plane is $12\,x-11\,y-z=6$, we also see that $\langle 12,-11,-1\rangle$ is normal to the tangent plane. These two normal vectors must point in the same direction or opposite directions. Hence, there is a number k such that

    $$\langle 12,-4y_0^3-z_0,-y_0\rangle = k\,\langle 12,-11,-1\rangle.$$

    Comparing first components, we see that $12=12k$ so that $k=1$. Comparing the other two components (and taking $k=1$), we obtain

    $$-4y_0^3-z_0=-11 \quad \text{and} \quad -y_0=-1.$$

    The second of these gives $y_0=1$. Therefore the first equation gives $-4-z_0=-11$ so $z_0=7$. We deduce that $A=(2,y_0,z_0)=(2,1,7)$. (As a check, note that $2^3=1^4+1\cdot 7$ so $(2,1,7)$ is a point on the surface $x^3=y^4+yz$.) ■

19. $x=2+t,\ y=3+t,\ z=-1-t$

20. $(0,1,0)$ and $(0,-1,0)$

21. $x=4+t,\ y=-1-t,\ z=2+2t$

22. $x=1-t,\ y=1+t,\ z=-1-t$

23. (a) See work below. The distances should be $z_0 + 1$ in both cases.

(b) $\langle x_0, y_0, -2\rangle$

(c) See below.

(d) See below.

Solution. (a) Since $4z_0 = x_0^2 + y_0^2$, we see that $z_0 \geq 0$. The distance from P to the point $(0, 0, 1)$ is

$$\sqrt{x_0^2 + y_0^2 + (z_0 - 1)^2} = \sqrt{x_0^2 + y_0^2 + z_0^2 - 2z_0 + 1}.$$

Since $4z_0 = x_0^2 + y_0^2$, we can rewrite this distance as

$$\sqrt{4z_0 + z_0^2 - 2z_0 + 1} = \sqrt{z_0^2 + 2z_0 + 1} = \sqrt{(z_0^2 + 1)^2} = z_0 + 1,$$

where for this last equation we used that $z_0 \geq 0$. The distance from $P = (x_0, y_0, z_0)$ to the plane $z = -1$ is simply the distance from (x_0, y_0, z_0) to $(x_0, y_0, -1)$ which is $z_0 + 1$. Comparing, we see that the distance from P to the point $(0, 0, 1)$ is the same as the distance from P to the plane $z = -1$.

(b) The surface $4z = x^2 + y^2$ can be written as $F(x, y, z) = 0$ where $F(x, y, z) = x^2 + y^2 - 4z$. The gradient at $P = (x_0, y_0, z_0)$ is perpendicular to the tangent plane at P. Since the gradient is $\nabla F(x, y, z) = \langle 2x, 2y, -4\rangle$, we see that the gradient at P is $\langle 2x_0, 2y_0, -4\rangle = 2\langle x_0, y_0, -2\rangle$. So the vector $\langle x_0, y_0, -2\rangle$ is perpendicular to the tangent plane at P.

(c) We have

$$\overrightarrow{n} = \langle x_0, y_0, -2\rangle, \quad \overrightarrow{u} = \langle -x_0, -y_0, 1 - z_0\rangle \quad \text{and} \quad \overrightarrow{v} = \langle 0, 0, 1\rangle.$$

The vector $\overrightarrow{m} = \langle y_0, -x_0, 0\rangle$, with components coming from the coefficients of the plane $y_0x - x_0y = 0$, is perpendicular to that plane. To show that the vectors $\overrightarrow{n}$, $\overrightarrow{u}$ and $\overrightarrow{v}$ are parallel to or on the plane, it suffices to show that these three vectors are all perpendicular to $\overrightarrow{m}$. Directly checking, we see that

$$\overrightarrow{m} \cdot \overrightarrow{n} = \overrightarrow{m} \cdot \overrightarrow{u} = \overrightarrow{m} \cdot \overrightarrow{v} = 0.$$

Therefore, the vectors $\overrightarrow{n}$, $\overrightarrow{u}$ and $\overrightarrow{v}$ are all parallel to or on the plane $y_0x - x_0y = 0$.

(d) Let θ_1 be the the smallest angle between vectors $\overrightarrow{n}$ and $\overrightarrow{u}$, and let θ_2 be the smallest angle between vectors $\overrightarrow{n}$ and $\overrightarrow{v}$. We want to show that $\theta_1 = \theta_2$. Since $4z_0 = x_0^2 + y_0^2$, we obtain

$$\overrightarrow{n} \cdot \overrightarrow{u} = -x_0^2 - y_0^2 - 2 + 2z_0 = -4z_0 - 2 + 2z_0 = -2(z_0 + 1).$$

Also, from part (a), we have $\sqrt{x_0^2 + y_0^2 + (z_0 - 1)^2} = z_0 + 1$, so

$$\|\overrightarrow{u}\| = \sqrt{x_0^2 + y_0^2 + (1 - z_0)^2} = z_0 + 1.$$

Hence,

$$\cos\theta_1 = \frac{\overrightarrow{n} \cdot \overrightarrow{u}}{\|\overrightarrow{n}\|\,\|\overrightarrow{u}\|} = \frac{-2(z_0 + 1)}{\|\overrightarrow{n}\|\,(z_0 + 1)} = \frac{-2}{\|\overrightarrow{n}\|}.$$

Since $\overrightarrow{n} \bullet \overrightarrow{v} = -2$ and $\|\overrightarrow{v}\| = 1$, we also have

$$\cos\theta_2 = \frac{\overrightarrow{n} \bullet \overrightarrow{v}}{\|\overrightarrow{n}\|\,|\overrightarrow{u}\|} = \frac{-2}{\|\overrightarrow{n}\|}.$$

Since θ_1 and θ_2 are both in the interval $[0, \pi]$ (because the smallest angle between two vectors is always in this interval) and since $\cos\theta_1 = \cos\theta_2$, we see that $\theta_1 = \theta_2$. ■

---

§15. Homework Set 15: Maxima and Minima

FT 1. For the function $f(x, y) = x^2 + x^2y^2 + x^4 + 2y^2 + 8y$, there is one point (x, y) where there is a local maximum, a local minimum, or a saddle point. Find this point (x, y), and determine whether it is the location of a local maximum, a local minimum or a saddle point.

FQ 2. There is one point $P = (x, y)$ where the function

$$f(x, y) = 2x^2 - xy + 3y + x - 4.$$

has a relative maximum, a relative minimum or a saddle point. Find the point P and indicate whether there is a relative maximum, a relative minimum or a saddle point there.

FT 3. The function $f(x, y) = x^3 + 3y^2 - 6xy$ has two critical points. Find them, and determine whether each is the location of a relative maximum, a relative minimum or a saddle point.

FT 4. The function

$$f(x, y) = 2x^2 - 4y^2 + x^2y^2 + 12x$$

has three critical points (x, y). Find them, and determine whether each is the location of a relative maximum, a relative minimum or a saddle point.

FT 5. The function $f(x, y) = x^2y + 4xy + 3y^3 - 5y$ has four critical points. Find them, and determine whether each is the location of a relative maximum, a relative minimum or a saddle point.

FT 6. The function $f(x, y) = 2x^2 + y^2 + 2xy^2$ has three critical points (a, b). Find them, and indicate below whether each is the location of a relative maximum, a relative minimum or a saddle point.

FT 7. Let

$$f(x, y) = x^4 + 4xy + xy^2.$$

The function $f(x, y)$ has 3 critical points. Calculate the three critical points and indicate (with justification) whether each determines a local maximum value of $f(x, y)$, a local minimum value of $f(x, y)$, or a saddle point of $f(x, y)$.

FQ 8. Find all relative maxima and minima *values* for the function

$$f(x,y) = 4x^2e^y - 2x^4 - e^{4y}$$

and where they occur. Indicate whether each value is a relative maximum value or a relative minimum value.

FF 9. Let $f(x,y) = x^2y^3 - 8x^2 - 12y^2$. There are 3 critical points. Find them, and determine whether each critical point is the location of a local maximum, a local minimum or a saddle point (so you are to choose one of three possibilities for each point). Be sure to justify your answers.

FT 10. Let

$$f(x,y) = (3y^4 + 1)(x^2 - 2x + 2) - 12y^3 + 12y^2.$$

The function $f(x,y)$ has 3 critical points. Calculate the critical points and indicate (with justification) whether each determines a local maximum value of $f(x,y)$, a local minimum value of $f(x,y)$, or a saddle point of $f(x,y)$.

FF 11. Let $f(x,y) = 2x^3y - 3x^2y - 5y^2$. The function $f(x,y)$ has two critical points other than $(0,0)$. Find these two critical points and determine whether each determines the location of a local maximum, a local minimum, or a saddle point.

FT 12. Let

$$f(x,y) = 2x^2y - 8xy + y^2 + 5.$$

The function $f(x,y)$ has 3 critical points. Calculate the critical points and indicate (with justification) whether each determines a local maximum value of $f(x,y)$, a local minimum value of $f(x,y)$, or a saddle point of $f(x,y)$.

FT 13. The function $f(x,y) = x^2y^2 + 2xy + y^2 - 2y$ has two critical points. Find them, and determine whether each is the location of a relative maximum, a relative minimum or a saddle point. (Hint: You can write $f_y = 2x(xy+1) + 2y - 2$. Note that if $xy + 1 = 0$, then $f_y = 2y - 2$.)

FF 14. Let $f(x,y) = 3x^2y + 3x^2 - y^3$. Determine whether $(1,-1)$ is the location of a local maximum, a local minimum, a saddle point, or not a critical point.

FT 15. Using the second derivative test for functions of two variables, find all points (a,b,c) where the graph of $f(x,y) = x^2 + 2xy + 2y^2 + 2x + 1$ has a local maximum or a local minimum. For each such point, indicate which (a local maximum or a local minimum) occurs.

FF 16. Using the second derivative test for functions of two variables, find all points (a,b,c) where the graph of $f(x,y) = 3y^4 - 5y^2 + 2xy + x^2 + 3$ has a local maximum, a local minimum, or a saddle point. For each such point, indicate which (a local maximum, a local minimum, or a saddle point) occurs.

FT 17. The function $f(x,y) = x^2y^2 - 6x^3 - y^2 + 5x^2$ has four critical points (a,b). Using the information below to help with computations, find the four critical points, and indicate whether each is the location of a relative maximum, a relative minimum or a saddle point.

$$f_x = 2xy^2 - 18x^2 + 10x, \quad f_y = 2x^2y - 2y = 2y(x^2 - 1),$$

$$f_{xx} = 2y^2 - 36x + 10, \quad f_{yy} = 2x^2 - 2, \quad f_{xy} = 4xy$$

FT 18. Let

$$f(x, y) = xy(x + 2)\, e^{-2y^2}.$$

Then

$$f_x = 2y(x + 1)\, e^{-2y^2}, \qquad f_y = -x(x + 2)(2y - 1)(2y + 1)\, e^{-2y^2}$$

$$f_{xx} = 2y\, e^{-2y^2}, \qquad f_{yy} = 4xy(x + 2)(4y^2 - 3)\, e^{-2y^2},$$

and

$$f_{xy} = -2(x + 1)(2y - 1)(2y + 1)\, e^{-2y^2}.$$

The function $f(x, y)$ has four critical points. Calculate the four critical points and indicate (with justification) whether each determines a local maximum value of $f(x, y)$, a local minimum value of $f(x, y)$, or a saddle point of $f(x, y)$.

FF 19. Let $f(x, y) = x^2y - xy^2 + 3xy$. Determine whether each point below is the location of a local maximum, a local minimum, a saddle point, or not a critical point (so you are to choose one of four possibilities for each point).

(a) $(0, 0)$ (b) $(0, 3)$ (c) $(1, -1)$ (d) $(-1, 1)$ (e) $(-3, 0)$

FT 20. Let

$$f(x, y) = y\,(y^2 - 3)\,(x + 1)\,(x - 2),$$

and consider the points

$$P = (-1, 0), \quad Q = (1/2, 1), \quad \text{and} \quad R = (1/2, -1).$$

These points may or may not be critical points for $f(x, y)$. Decide whether each is the location of a local maximum value of $f(x, y)$, a local minimum value of $f(x, y)$, a saddle point of $f(x, y)$, or none of these (i.e., not a critical point). So you have four choices for each point. Be sure to fully justify your answers. Below is some (not all that is needed) information on $f(x, y)$ to help with your calculations.

$$f_{xx}(x, y) = 2\,y\left(y^2 - 3\right), \quad f_{yy}(x, y) = 6\,y\,(x + 1)\,(x - 2)\,,$$

$$f_{x,y}(x, y) = 3\,(y - 1)\,(y + 1)\,(2\,x - 1)\,.$$

FT 21. Let

$$f(x, y) = x^2y^2 + xy^3 + 3y^2 + 5y.$$

Then

$$f_x = y^2(y + 2x) \quad \text{and} \quad f_y = 2x^2y + 3xy^2 + 6y + 5.$$

Also,
$$f_{xx} = 2y^2, \quad f_{yy} = 2x^2 + 6xy + 6, \quad \text{and} \quad f_{xy} = 3y^2 + 4xy.$$
You may use the above information to help with this problem. There are three critical points, each a point (x_0, y_0) that is a location of a local maximum, a local minimum or a saddle point. One of these has a value of x_0 that is ± 2, ± 1 or $\pm 1/2$ (one of those 6 numbers). Find this critical point, and determine whether it is a location of a local maximum, a local minimum, or a saddle point.

FQ 22. The function
$$f(x, y) = 5 - x^3 + 27x + y^2 - 4y$$
has two critical points. Find the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ where the critical points occur, and determine whether there is a local maximum, a local minimum or a saddle point at each of the points.

FT 23. Let
$$f(x, y) = 12\,xy^2 - x^4 - 8\,xy^3,$$
so that
$$f_x(x, y) = 12\,y^2 - 4\,x^3 - 8\,y^3 \quad \text{and} \quad f_y(x, y) = 24\,xy - 24\,xy^2.$$
Let
$$P = (-1, 1), \quad Q = (1, 1), \quad R = (-1, 0) \quad \text{and} \quad S = (0, 3/2).$$
These four points may or may not be critical points for $f(x, y)$. Decide whether each is the location of a local maximum value of $f(x, y)$, a local minimum value of $f(x, y)$, a saddle point of $f(x, y)$, or none of these (i.e., not a critical point). So you have four choices for each point. Be sure to fully justify your answers.

FT 24. Determine the maximum and minimum values of $f(x, y) = 3x + xy^2$ on $S = \{(x, y) : x^2 + y^2 \leq 9\}$ as well as all points (x, y) where these extreme values occur. Note this is a problem about global extrema.

FQ 25. Let
$$f(x, y) = x^3 + 2xy^2 - 5x^2 \quad \text{and} \quad D = \{(x, y) : x^2 + y^2 \leq 4\}.$$
(a) Find all points (x, y) where $f_x = 0$ and $f_y = 0$. You should get two points. One will be inside D, and one will be outside D.

(b) Determine the absolute maximum value and the absolute minimum value of $f(x, y)$ on the boundary of D, where $x^2 + y^2 = 4$.

(c) Determine the absolute maximum value and the absolute minimum value of $f(x, y)$ on all of D, where $x^2 + y^2 \leq 4$. Include values of $f(x, y)$ inside and on the boundary of D.

FQ 26. Determine the absolute maximum *value* and absolute minimum *value* of
$$f(x, y) = (x + 1)\,y^2 + x^2 - x$$

where (x, y) is a point satisfying

$$x^2 + y^2 \leq 4.$$

Furthermore, determine all points (x, y) as above where the absolute maximum value and absolute minimum value occur.

FF 27. Determine the maximum and the minimum values of the function

$$18x^2 - 6x + 3 - 24xy + 16y^2$$

on the triangle $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x\}$. Simplify your answers.

FF 28. Let $f(x, y) = 5x^2 + 2xy + 2y^2 - 4x - 2y + 7$. Determine the absolute maximum value and absolute minimum value of $f(x, y)$ on the square $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Be sure to completely justify your answers.

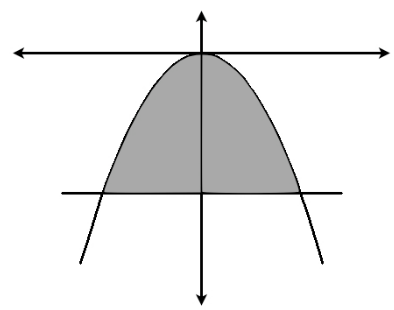
FF 29. Let

$$f(x, y) = x^2y + x^2 - 2xy + y^2,$$

and let

$$R = \{(x, y) : -2 \leq y \leq -x^2\}.$$

Thus, R is the set of points below $y = -x^2$ and above $y = -2$. Find the absolute maximum value of $f(x, y)$ and the absolute minimum value of $f(x, y)$ on R (including points inside R and on the boundary of R). Indicate all points you are considering for the location of the absolute maximum and absolute minimum values.



FT 30. (a) Determine the absolute maximum and absolute minimum value of

$$f(x, y) = 3x^2 + 6y^2 - 2x$$

where (x, y) varies over the points satisfying $x^2 + y^2 \leq 1$. Furthermore, indicate ALL points (x, y) satisfying $x^2 + y^2 \leq 1$ where these values occur.

(b) For this problem, you may want to use the work you already did above. Determine the absolute maximum and absolute minimum value of

$$f(x, y) = 3x^2 + 6y^2 - 2x$$

where (x, y) varies over the points satisfying $x^2 + y^2 = 1$. Furthermore, indicate ALL points (x, y) on $x^2 + y^2 = 1$ where these values occur.

FT 31. For both parts of this problem, consider

$$f(x, y) = 9x^2 + 6y^2 + 6x + 4.$$

(a) Determine the global maximum and the global minimum value of $f(x, y)$ on the circle $x^2 + y^2 = 4$.

(b) Let $R = \{(x, y) : x^2 + y^2 \leq 4\}$, so R is the circle centered at the origin of radius 2 together with its interior. Determine the global maximum and the global minimum value of $f(x, y)$ on R.

FQ 32. (a) Determine the absolute maximum and absolute minimum value of

$$f(x, y) = 3x^2 + 2y^2 - 3x - 4$$

where (x, y) varies over the points satisfying $x^2 + y^2 \leq 4$. Furthermore, indicate ALL points (x, y) satisfying $x^2 + y^2 \leq 4$ where these values occur.

(b) For this problem, you should only have to use the work you already did above. Determine the absolute maximum value and the absolute minimum value of

$$f(x, y) = 3x^2 + 2y^2 - 3x - 4$$

where (x, y) varies over the points satisfying $x^2 + y^2 = 4$. Furthermore, indicate ALL points (x, y) satisfying $x^2 + y^2 = 4$ where these values occur.

FT 33. (a) Determine the absolute maximum and absolute minimum value of

$$f(x, y) = x^4 + x^2y^2 + y^2 - 4x$$

where (x, y) varies over the points satisfying $x^2 + y^2 \leq 4$. Furthermore, indicate ALL points (x, y) satisfying $x^2 + y^2 \leq 4$ where these values occur.

(b) For this problem, you may want to use the work you already did above. Determine the absolute maximum value and the absolute minimum value of

$$f(x, y) = x^4 + x^2y^2 + y^2 - 4x$$

where (x, y) varies over the points satisfying $x^2 + y^2 = 4$. You do **NOT** need to indicate the points (x, y) on $x^2 + y^2 = 4$ where these values occur.

FT 34. Determine the absolute maximum and absolute minimum value of

$$f(x, y) = 4x^3 - 9x^2 - 18y^2 - 12x + 21$$

where (x, y) varies over the points satisfying $x^2 + y^2 \leq 1$. Furthermore, indicate ALL points (x, y) satisfying $x^2 + y^2 \leq 1$ where these values occur.

FT 35. Determine the absolute maximum and absolute minimum of

$$f(x, y) = 4xy^2 - 4x$$

where (x, y) is a point satisfying

$$x^2 + 4y^2 \leq 16.$$

Furthermore, determine all points (x, y) as above where the absolute maximum and absolute minimum occur.

FQ 36. Determine the points (x, y) where the global maximum and the global minimum occur for the function

$$f(x, y) = 4x^3 + 4x^2 + 3y^2 - 4x$$

on the set

$$S = \{(x, y) : x^2 + y^2 \leq 1\}.$$

Comment: If you do this problem as intended, you should be factoring two quadratics. Both of the quadratics factor "nicely".

FT 37. Determine the absolute maximum and absolute minimum of

$$f(x, y) = x^4 - 4y^2 - 18x^2 + 5$$

where (x, y) is a point satisfying

$$\frac{x^2}{4} + \frac{y^2}{16} \leq 1.$$

In other words, (x, y) is on or inside the ellipse $(x^2/4) + (y^2/16) = 1$. Furthermore, determine all points (x, y) as above where the absolute maximum and absolute minimum occur.

FT 38. Determine the absolute maximum value and absolute minimum value of

$$f(x, y) = 8\,x^3 + 9\,x^2 + 9\,y^2 - 6\,x$$

where (x, y) varies over the points satisfying $x^2 + y^2 \leq 1$ (a circle of radius 1 and its interior). Furthermore, indicate ALL points (x, y) satisfying $x^2 + y^2 \leq 1$ where these values occur.

FT 39. Find the maximum and minimum values for the function $f(x, y) = xy^2 + 3y^2 + 5x - 5$ in the disk $x^2 + y^2 \leq 4$. Be sure you justify your answers.

FQ 40. Find the absolute extreme *values* of $f(x, y) = 2x^2 + y^2 - 2x - 3$ on the disk R of points (x, y) satisfying $x^2 + y^2 \leq 4$.

FT 41. Let $R = \{(x, y) : x^2 + y^2 \leq 4\}$, so R is the circle centered at the origin of radius 2 together with its interior. Let $f(x, y)$ be defined on R by

$$f(x, y) = 6x^2 + 3y^2 - 6x - 9.$$

Calculate the global maximum value and the global minimum value of $f(x, y)$ on R.

FT 42. For this problem, let

$$f(x, y) = 2x^3 + 5x^2 + 4xy^2,$$

and let $R = \{(x, y) : x^2 + y^2 \leq 4\}$.

(a) What are the global maximum and minimum values of $f(x, y)$ on the boundary of R, that is on the circle $x^2 + y^2 = 4$? Justify your answers.

(b) What are the global maximum and minimum values of $f(x, y)$ for all points in R, both inside and on the circle $x^2 + y^2 = 4$? Justify your answers. You should obtain two points in the interior of R that need to be considered for this part in addition to the points that you found in part (a).

FT 43. For this problem,

$$f(x,y) = (3x^2 + 9x)(x^2 + y^2) - 4x^3 \quad \text{and} \quad S = \{(x,y) : x^2 + y^2 \le 4\}.$$

To calculate the absolute maximum and absolute minimum of $f(x,y)$ on S, normally you want to calculate the critical points of $f(x,y)$ inside S (where $x^2 + y^2 < 4$). However, in this case, your teacher has already done that. He found that the critical points inside S are $(0,0)$ and $(-5/4, 0)$. He also calculated that

$$f(0,0) = 0 \quad \text{and } f(-5/4, 0) = -\frac{625}{256}.$$

Unfortunately, your teacher didn't check points on the boundary of S (where $x^2+y^2 = 4$), so he is not sure what the absolute maximum value and the absolute minimum value of $f(x,y)$ on S is. Determine the absolute maximum value and the absolute minimum value of $f(x,y)$ on S.

FF 44. The function

$$f(x,y) = x^2 + 2y^2 - 2xy - 4y + 3$$

is defined for all points (x,y) in the plane. Explain why $f(x,y)$ has an absolute maximum value or an absolute minimum value and calculate that value. Justify your answer.

FQ 45. Calculate the minimum distance from the point $(1,2,1)$ to a point on the hyperboloid of one sheet given by

$$\frac{x^2}{2} - y^2 + (z-1)^2 = 1.$$

FF 46. Let ℓ denote the line given by the parametric equations $x = 1 - t$, $y = -1 - t$, and $z = 2$ (so any value of t produces a point (x,y,z) on line ℓ). Let ℓ' denote the line given by the parametric equations $x = -1 + s$, $y = 3 + s$, and $z = -1 + 3s$ (so any value of s produces a point (x,y,z) on line ℓ'). The purpose of this problem is to have you determine the point P on ℓ and the point Q on ℓ' with the distance from P to Q as small as possible. This distance is the distance between two lines. But this problem is not asking for that distance. Instead, we are finding the points P and Q.

(a) Write a function D of t and s that represents the square of the distance from a point on the line ℓ to a point on the line ℓ'.

(b) Find the values of t and s that minimize the distance from a point on the line ℓ to a point on the line ℓ'.

(c) What are the points P and Q mentioned above before part (a)?

FF 47. Let ℓ_1 and ℓ_2 be two lines given by parametric equations as follows:

$$\ell_1 : \begin{cases} x = t \\ y = 2t + 1 \\ z = t \end{cases} \qquad \ell_2 : \begin{cases} x = t \\ y = 3t \\ z = -t \end{cases}$$

The lines ℓ_1 and ℓ_2 are skew. You do not need to show this. Find the point P on ℓ_1 and the point Q on ℓ_2 such that the distance from P to Q is as small as possible (i.e., the distance from P and Q is the distance between the two lines). One way to approach this problem is to minimize a function in two variables (one that you have to figure out).

FF 48. The curve C_1 is given by $x = 2t$, $y = t$ and $z = t + 2$ for $t \in (-\infty, \infty)$. The curve C_2 is given by $x = s$, $y = -2s$ and $z = 6s^2$ for $s \in (-\infty, \infty)$. Find a point P_1 on C_1 and a point P_2 on C_2 such that the distance from P_1 to P_2 is as small as possible. (Hints: You can use the information in this hint to help determine a correct answer to the problem. There are two different correct answers, that is two different choices for the pair of points P_1 and P_2 that minimize this distance. You only need to give me one pair of points P_1 and P_2. None of the points in your answer should have an x coordinate equal to 0.)

FF 49. The purpose of this problem is to use material from this course to find the distance (that is, the minimum distance) between the graphs of $y = x^2$ and $y = 3x^2 + 1$ as well as the points on these graphs that produce this minimum distance. To do this we consider a point P on the graph of $y = x^2$ and a point Q on the graph of $y = 3x^2 + 1$. The point P will have the form (t, t^2) for some t and the point Q will have the form $(s, 3s^2 + 1)$ for some s.

(a) Taking $P = (t, t^2)$ and $Q = (s, 3s^2 + 1)$, write a formula for the *square* of the distance from P to Q.

(b) Let $f(s, t)$ be the function in part (a). Find the pairs (s, t) that minimize the value of the function f. There should be two. (Hint: Factor $f_s + f_t$.)

(c) What is the minimum distance between the graphs of $y = x^2$ and $y = 3x^2 + 1$?

(d) Find a point P on the graph of $y = x^2$ and a point Q on the graph of $y = 3x^2 + 1$ such that the distance PQ is the minimum distance given in part (c). There are two correct answers; you only need to give one of them.

FF 50. In this problem, $\mathcal{P}$ is the plane $4x - z = 3$. We begin with an arbitrary point $A = (x_0, y_0, z_0)$ and find a formula for the distance from A to the plane $\mathcal{P}$ by finding the point B on $\mathcal{P}$ that is closest to A. Then we find the point A on the paraboloid $z = 2x^2 + 3y^2$ that is closest to the plane $4x - z = 3$.

(a) What are parametric equations for the line that passes through the point $A = (x_0, y_0, z_0)$ and is perpendicular to the plane $\mathcal{P}$ given by $4x - z = 3$?

(b) If $B = (x_1, y_1, z_1)$ is the point on plane $\mathcal{P}$ that is closest to A, then explain why

$$x_1 - x_0 = \frac{4}{17}\big(3 - 4x_0 + z_0\big), \quad y_1 - y_0 = 0, \quad \text{and} \quad z_1 - z_0 = -\frac{1}{17}\big(3 - 4x_0 + z_0\big).$$

(Hint: The point B that is the intersection of the line in part (a) with the plane $\mathcal{P}$.)

(c) If $\operatorname{dist}(A, B)$ is the distance from the point A to the point B, explain why

$$\operatorname{dist}(A, B) = \frac{\big|3 - 4x_0 + z_0\big|}{\sqrt{17}}.$$

(d) What point $A = (x_0, y_0, z_0)$ on the paraboloid $z = 2x^2 + 3y^2$ is closest to the plane

$4x - z = 3$? (Hint: You want to minimize dist(A,B). Use that, since A is on the paraboloid, $z_0 = 2x_0^2 + 3y_0^2$.)

FF 51. You have dealt with a number of problems involving maximizing and minimizing functions in two variables over closed and bounded regions, where you had to consider both points in the interior of the region and points on the boundary of the region. In this problem, you are to do the same for a function of three variables. Let

$$f(x, y, z) = 8x^2 - xy^2 + y^2 - z^2.$$

Let S be the solid given by

$$S = \{(x, y, z) : x^2 + y^2 + z^2 \leq 5\}.$$

So S is the sphere of radius $\sqrt{5}$ centered at the origin together with its interior. The goal in this problem is to find the maximal and minimal values of $f(x, y, z)$ with (x, y, z) in S.

(a) The critical points inside S (and not on the boundary) are those points (x, y, z) inside S where each one of f_x, f_y and f_z is equal to 0. These are the possible points inside S where $f(x, y, z)$ can obtain its maximal or minimal value. Find the critical points inside S.

(b) The boundary points of S are those points (x, y, z) where $x^2 + y^2 + z^2 = 5$. Note that for all such (x, y, z) we have $x^2 + y^2 \leq 5$. Also, for such points, $z = \pm\sqrt{5 - x^2 - y^2}$ so that $f(x, y, z) = g(x, y)$ where

$$g(x, y) = f\left(x, y, \pm\sqrt{5 - x^2 - y^2}\right) = 8x^2 - xy^2 + y^2 - (5 - x^2 - y^2) = 9x^2 - xy^2 + 2y^2 - 5.$$

The points on the boundary of S that you should consider then correspond to (x, y) that maximize or minimize $g(x, y)$. Find all the points (x, y) in

$$R = \{(x, y) : x^2 + y^2 \leq 5\}$$

where $g(x, y)$ is maximal or minimal. (Be sure to consider both interior points to R and boundary points of R when doing this part.)

(c) Determine the maximum value and the minimum value of $f(x, y, z)$ for (x, y, z) in S (including boundary points).

Answers for §15

1. There is a local minimum at $(0, -2)$.

2. There is a saddle point at $(3, 13)$.

3. There is a relative minimum at $(2, 2)$ and a saddle point at $(0, 0)$.

4. There is a relative minimum at $(-3, 0)$ and saddle points at $(-2, \pm 1)$.

5. There is a relative maximum at $(-2, -1)$, a relative minimum at $(-2, 1)$ and saddle points at $(1, 0)$ and $(-5, 0)$.

6. There is a relative minimum at $(0, 0)$ and saddle points at $(-1/2, \pm 1)$.

7. There is a local minimum at $(1, -2)$ and saddle points at $(0, 0)$ and $(0, -4)$.

8. There is a relative maximum value of 1 at both of the points $(\pm 1, 0)$. There are no other relative extrema.

9. There is a local maximum at $(0, 0)$ and saddle points at $(\pm 2, 2)$.

10. There are local minima at $(1, 0)$ and $(1, 2)$ and a saddle point at $(1, 1)$.

11. There is a local maximum at $(1, -1/10)$ and a saddle point at $(3/2, 0)$.

12. There is a local minimum at $(2, 4)$ and saddle points at $(0, 0)$ and $(4, 0)$.

13. There is a relative minimum at $(-1, 1)$ and a saddle point at $(1, 0)$.

14. a saddle point

15. There is a local minimum at $(-2, 1, -1)$.

16. There are local minima at $(1, -1, 0)$ and $(-1, 1, 0)$ and a saddle point at $(0, 0, 3)$.

17. There is a relative maximum at $(5/9, 0)$ and saddle points at $(0, 0)$, $(1, 2)$ and $(1, -2)$.

18. There is a local maximum at $(-1, -1/2)$, a local minimum at $(-1, 1/2)$ and saddle points at $(0, 0)$ and $(-2, 0)$.

19. (a) a saddle point; (b) a saddle point; (c) not a critical point; (d) a local minimum; (e) a saddle point

20. P is the location of a saddle point; Q is the location of a local maximum; R is the location of a local mimimum

21. There is a local minimum at $(1/2, -1)$.

22. There is a local minimum at $(-3, 2)$ and a saddle point at $(3, 2)$.

23. P is not a critical point; Q is the location of a local maximum; R is not a critical point; S

is the location of a saddle point

24. The maximum value is 16 which occurs at $(2, \pm\sqrt{5}\,)$, and the minimum value is -16 which occurs at $(-2, \pm\sqrt{5}\,)$.

25. (a) $(0,0)$ and $(10/3, 0)$

    (b) The absolute maximum value on the boundary is $76/27$, and the absolute minimum value on the boundary is -28.

    (c) The absolute maximum value on all of D is $76/27$, and the absolute minimum value on all of D is -28.

26. The absolute maximum value of $f(x, y)$ is 6, and it occurs at $(1, \pm\sqrt{3}\,)$ and $(-2, 0)$. The absolute minimum value of $f(x, y)$ is $-1/4$, and it occurs at $(1/2, 0)$.

27. The absolute maximum value is 15 (occurring at $(1, 0)$), and the absolute minimum value is 2 (occurring at $(1/3, 1/4)$).

28. The absolute maximum value of $f(x, y)$ is 10, and it occurs at $(1, 1)$. The absolute minimum value of $f(x, y)$ is 6, and it occurs at $(1/3, 1/3)$.

29. The absolute maximum value of $f(x, y)$ is $2 + 4\sqrt{2}$, and it occurs at $(\sqrt{2}, -2)$. The absolute minimum value of $f(x, y)$ is $2 - 4\sqrt{2}$, and it occurs at $(-\sqrt{2}, -2)$.

30. (a) The absolute maximum value is $19/3$, and it occurs at $(-1/3, \pm 2\sqrt{2}/3)$. The absolute minimum value is $-1/3$, and it occurs at $(1/3, 0)$.

    (b) The absolute maximum value is $19/3$, and it occurs at $(-1/3, \pm 2\sqrt{2}/3)$. The absolute minimum value is 1, and it occurs at $(1, 0)$.

31. (a) The global maximum value is 52, and the global minimum value is 25.

    (b) The global maximum value is 52, and the global minimum value is 3.

32. (a) The absolute maximum value is 14, and it occurs at $(-2, 0)$. The absolute minimum value is $-19/4$, and it occurs at $(1/2, 0)$.

    (b) The absolute maximum value is 14, and it occurs at $(-2, 0)$. The absolute minimum value is $7/4$, and it occurs at $(3/2, \pm\sqrt{7}/2)$.

33. (a) The absolute maximum value is 24, and it occurs at $(-2, 0)$. The absolute minimum value is -3, and it occurs at $(1, 0)$.

    (b) The absolute maximum value is 24, and it occurs at $(-2, 0)$. The absolute minimum value is $8/3$, and it occurs at $(2/3, \pm 4\sqrt{2}/3)$.

34. The absolute maximum value is $97/4$, and it occurs at $(-1/2, 0)$. The absolute minimum

value is $-1/4$, and it occurs at $(1/2, \pm\sqrt{3}/2)$.

35. The absolute maximum value is 16, and it occurs at $(-4, 0)$ and $(2, \pm\sqrt{3}\,)$. The absolute minimum value is -16, and it occurs at $(4, 0)$ and $(-2, \pm\sqrt{3}\,)$.

36. The absolute maximum value is 133/27, and it occurs at $(-2/3, \pm\sqrt{5}/3)$. The absolute minimum value is $-20/27$, and it occurs at $(1/3, 0)$.

37. The absolute maximum value is 5, and it occurs at $(0, 0)$. The absolute minimum value is -60, and it occurs at $(\pm 1, \pm 2\sqrt{3})$.

38. The absolute maximum value is 11, and it occurs at $(1, 0)$ and $(-1/2, \pm\sqrt{3}/2)$. The absolute minimum value is $-13/16$, and it occurs at $(1/4, 0)$.

39. The (absolute) maximum value is 12 (occurring at $(1, \pm\sqrt{3}\,)$). The (absolute) minimum value is -15 (occurring at $(-2, 0)$).

40. The absolute maximum value is 9 (occurring at $(-2, 0)$). The absolute minimum value is $-7/2$ (occurring at $(1/2, 0)$).

41. The absolute maximum value is 27 (occurring at $(-2, 0)$). The absolute minimum value is $-21/2$ (occurring at $(1/2, 0)$).

42. (a) The global maximum value is 36 (occurring at $(2, 0)$). The global minimum value is -9 (occurring at $(-1, \pm\sqrt{3}\,)$).

    (b) The global maximum value is 36 (occurring at $(2, 0)$). The global minimum value is -9 (occurring at $(-1, \pm\sqrt{3}\,)$).

43. The absolute maximum value is 88 (occurring at $(2, 0)$). The absolute minimum value is -20 (occurring at $(-1, \pm\sqrt{3}\,)$).

44. One can rewrite $f(x, y)$ as $f(x, y) = (x-y)^2 + (y-2)^2 - 1$. Since $(x-y)^2 \geq 0$ and $(y-2)^2 \geq 0$ for all real values of x and y, we know $f(x, y) \geq -1$ for all real x and y. Since $f(2, 2) = -1$, there is an absolute minimum value of -1, occurring at $(2, 2)$.

45. $\sqrt{2}$

    Solution. We want to minimize the square of the distance from (x, y, z) on the hyperboloid of one sheet to $(1, 2, 1)$. Note that (x, y, z) on the hyperboloid of one sheet implies that

    $$(z-1)^2 = 1 - \frac{x^2}{2} + y^2.$$

    So we want to minimize the value of

    $$(x-1)^2 + (y-2)^2 + (z-1)^2 = (x-1)^2 + (y-2)^2 + 1 - \frac{x^2}{2} + y^2 = \frac{x^2}{2} + 2y^2 - 2x - 4y + 6.$$

Call this last function $f(x, y)$. Setting the partial derivatives to zero, we deduce that

$$x - 2 = 0 \quad \text{and} \quad 4y - 4 = 0,$$

so the point $(2, 1)$ is the unique point critical point of $f(x, y)$. On the other hand, we know that there must be a minimum distance from a point to a surface, so we know $f(x, y)$ has a minimum value. Since $(2, 1)$ is the only place where that minimum can occur, the minimum must be $f(2, 1) = 2$. Since $f(2, 1)$ is the square of the minimum distance we want, the minimum distance is $\sqrt{2}$. ■

46. (a) $(2 - t - s)^2 + (-4 - t - s)^2 + (3 - 3s)^2$

    (b) $t = -2$ and $s = 1$

    (c) $(3, 1, 2)$ on ℓ and $(0, 4, 2)$ on ℓ'

47. $P = (-2/15, 11/15, -2/15)$ and $Q = (1/5, 3/5, -1/5)$

48. $P_1 = (-1/6, -1/12, 23/12)$ and either $Q_1 = (1/2, -1, 3/2)$ or $Q_1 = (-1/2, 1, 3/2)$

49. (a) $(t - s)^2 + (t^2 - 3s^2 - 1)^2$

    (b) $(1, 1/3)$ and $(-1, -1/3)$

    (c) $\sqrt{5}/3$

    (d) The pair $P = (1, 1)$ and $Q = (1/3, 4/3)$ or the pair $P = (-1, 1)$ and $Q = (-1/3, 4/3)$.

50. (a) $x = x_0 + 4t, \;\; y = y_0, \;\; z = z_0 - t$

    (b) The closest point to A on the plane $\mathcal{P}$ is the intersection of the line in (a) and the plane. For the value of t that gives this point, we have $4(x_0 + 4t) - (z_0 - t) = 3$. Simplifying gives $4x_0 - z_0 + 17t = 3$, and so $t = (z_0 - 4x_0 + 3)/17$. Thus, $B = (x_1, y_1, z_1)$ is the point on the line in part (a) with $t = (z_0 - 4x_0 + 3)/17$. This gives

$$x_1 = x_0 + 4(z_0 - 4x_0 + 3)/17, \quad y_1 = y_0 \quad \text{and} \quad z_1 = z_0 - (z_0 - 4x_0 + 3)/17.$$

    This can be rewritten as

$$x_1 - x_0 = \frac{4}{17}\big(3 - 4x_0 + z_0\big), \quad y_1 - y_0 = 0, \quad \text{and} \quad z_1 - z_0 = -\frac{1}{17}\big(3 - 4x_0 + z_0\big),$$

    which is what was to be explained.

    (c) From part (b), the distance from A to B is

$$\begin{aligned}
&\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2} \\
&\qquad = \sqrt{\left(\frac{4}{17}\right)^2 \big(3 - 4x_0 + z_0\big)^2 + \left(-\frac{1}{17}\right)^2 \big(3 - 4x_0 + z_0\big)^2} \\
&\qquad = \sqrt{\left(\frac{16 + 1}{17^2}\right) \big(3 - 4x_0 + z_0\big)^2} = \frac{\sqrt{\big(3 - 4x_0 + z_0\big)^2}}{\sqrt{17}} = \frac{\big|3 - 4x_0 + z_0\big|}{\sqrt{17}},
\end{aligned}$$

which is what we wanted to show.

(d) Since $A = (x_0, y_0, z_0)$ is on the paraboloid, $z_0 = 2x_0^2 + 3y_0^2$. From part (c) then, the minimum distance is when the value of

$$(3 - 4x_0 + z_0)^2 = (3 - 4x_0 + 2x_0^2 + 3y_0^2)^2$$

is as small as possible. Since

$$3 - 4x_0 + 2x_0^2 + 3y_0^2 = 2(x_0 - 1)^2 + 3y_0^2 + 1$$

and since squares are always ≥ 0, the minimum of this expression is 1 when $x_0 = 1$ and $y_0 = 0$. Then $z_0 = 2x_0^2 + 3y_0^2 = 2$. Hence, the point A on the paraboloid $z = 2x^2 + 3y^2$ which is closest to the plane $4x - z = 3$ is $(1, 0, 2)$.

51. (a) The only critical point inside S is $(0, 0, 0)$.

(b) The points $(\pm\sqrt{5}, 0)$ give the maximal value 40 of $g(x, y)$. The point $(0, 0)$ gives the minimal value -5 of $g(x, y)$. The points $(1/3, \pm 2\sqrt{11}/3)$ should also be considered.

(c) The maximal value is 40 (occurring at $(\pm\sqrt{5}, 0, 0)$ in S). The minimal value is -5 (occurring at $(0, 0, \pm\sqrt{5})$ in S).

---

§16. Homework Set 16: Lagrange Multipliers

FF 1. Using Lagrange multipliers, determine the absolute maximum value and the absolute minimum value of $f(x, y) = 3x^2y + 3x^3 + 2y^3$ given the constraint $x^2 + y^2 = 1$.

FT 2. Using the method of Lagrange multipliers, determine the maximum and minimum values of $f(x, y) = xy$ given the constraint $4x^2 + y^2 = 8$.

FF 3. Using Lagrange multipliers, find the maximum value and the minimum value of $f(x, y) = x^3y$ given the constraint $3x^4 + y^4 = 1$.

FT 4. Using the method of Lagrange multipliers, determine the minimum value of $f(x, y) = x^2 + y^2 - x + y - 1$ given the constraint $x - y + 1 = 0$.

FT 5. Using a Lagrange multiplier, find the maximum and minimum values of the function

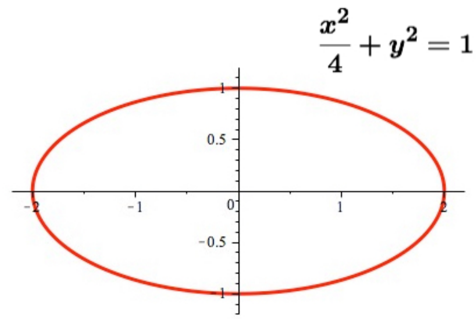
$$x^2 + 8y + 8z$$

given the constraint that (x, y, z) satisfies

$$x^2 + 2y^2 + 4z^2 = 48.$$

FT 6. Using the method of Lagrange multipliers, find the maximum and minimum values of the function $f(x,y) = y - x^2$ given the constraint $\dfrac{x^2}{8} + \dfrac{y^4}{4} = 1$. (Comment: You can avoid some arithmetic if you note that obtaining a "value" for $f(x,y)$ requires that you know the value of x^2, not the value of x.)

FT 7. (a) Using the method of Lagrange multipliers, find the maximum and minimum values of the function $f(x,y) = xy$ given the constraint $(x^2/4) + y^2 = 1$. In other words, calculate the maximum and minimum values of $f(x,y) = xy$ where (x,y) is restricted to points on the ellipse shown to the right. Be sure to show all the steps needed to get to your answers, and simplify both answers.



(b) Find the absolute maximum value and the absolute minimum value of $f(x,y) = xy$ in the set $S = \{(x,y) : (x^2/4) + y^2 \le 1\}$. In other words, find the absolute maximum value and the absolute minimum value of $f(x,y) = xy$ for (x,y) on the boundary or inside the ellipse shown in the part (a).

FT 8. Using Lagrange multipliers, calculate the distance from the origin to the nearest point on the line $x - 3y = 2$. Note that this problem is to be done with Lagrange multipliers even though you should be able to do find the distance other ways.

Answers for §16

1. The absolute maximum is $11/\sqrt{10}$ (occurring at $(3/\sqrt{10}, 1/\sqrt{10})$). The absolute minimum is $-11/\sqrt{10}$ (occurring at $(-3/\sqrt{10}, -1/\sqrt{10})$).

2. The maximum value is 2 (occurring at $(1,2)$ and $(-1,-2)$). The minimum value is -2 (occurring at $(1,-2)$ and $(-1,2)$).

3. The maximum value is $1/4$ (occurring at $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$). The minimum value is $-1/4$ (occurring at $(1/\sqrt{2}, -1/\sqrt{2})$ and $(-1/\sqrt{2}, 1/\sqrt{2})$).

4. The minimum value is $1/2$ (occurring at $(-1/2, 1/2)$). (There is no maximum.)

5. The maximum value is 60 (occurring at $(\pm 6, 2, 1)$). The minimum value is -48 (occurring at $(0, -4, -2)$).

6. The maximum value is $\sqrt{2}$ (occurring at $(0, \sqrt{2})$). The minimum value is $-67/8$ (occurring at $(\pm 3\sqrt{14}/4, -1/2)$).

7. (a) The maximum value is 1 (occurring at $(\sqrt{2}\,, 1/\sqrt{2}\,)$ and $(-\sqrt{2}\,, -1/\sqrt{2}\,)$). The minimum value is -1 (occurring at $(\sqrt{2}\,, -1/\sqrt{2}\,)$ and $(-\sqrt{2}\,, 1/\sqrt{2}\,)$).

   (b) The maximum value is 1 (occurring at $(\sqrt{2}\,, 1/\sqrt{2}\,)$ and $(-\sqrt{2}\,, -1/\sqrt{2}\,)$). The minimum value is -1 (occurring at $(\sqrt{2}\,, -1/\sqrt{2}\,)$ and $(-\sqrt{2}\,, 1/\sqrt{2}\,)$). (There is only the critical point $(0,0)$ inside the ellipse and $f(0,0) = 0$.)

8. The distance is $\sqrt{2/5}$ (from $(1/5, -3/5)$ on the line to the origin).