Calculus III: Homework Problem Sets[1]

Part I

§1. Homework Set 1: Introduction to 3-Dimensions

FT 1. Calculate the distance from the point $P = (0, 1, 2)$ to the point $Q = (3, 1, -2)$.

FT 2. Calculate the midpoint of the line segment $\overline{XY}$ where $X = (4, 3, 5)$ and $Y = (2, 5, -3)$.

FQ 3. What is the distance from the point $(-4, 5, -5)$ to the x-axis?

FT 4. What is the distance from $(3, 4, 5)$ to the x-axis?

FQ 5. Find the distance from the point $(-5, 2, -3)$ to the y-axis.

FT 6. What is the distance from $(3, 4, 6)$ to the z-axis?

FQ 7. Find the distance from the point $(3, 7, -4)$ to each of the following.
(a) the xy-plane
(b) the y-axis

FQ 8. The point $(2, y, z)$ is on a line through $(-3, 4, -2)$ that is parallel to one of the coordinate axes. Which axis must it be and what are y and z?

FQ 9. Suppose that a box has its faces parallel to the coordinate planes and that the points $(-5, 3, 2)$ and $(-1, 1, 2)$ are two endpoints of a diagonal *on one face* of the box. What are the two endpoints of the other diagonal on this same face of the box?

[1] FQ = Filaseta Quiz problem, FT = Filaseta Test problem, FF = Filaseta Final Exam problem

FQ 10. Complete the square to find the center and radius of the sphere whose equation is given by $4x^2 + 4y^2 + 4z^2 - 4x + 8y + 16z - 13 = 0$.

FQ 11. What is the radius of the sphere $x^2 + y^2 + z^2 = 12x + 6y - 4z$? Simplify your answer.

FT 12. The points satisfying $x^2 + y^2 + z^2 - 4x + 6z + 12 = 0$ represent either a point, a sphere, a cube or no graph. Decide which it is. If you answer a point, then determine what point it is. If you answer a sphere, determine the center and radius. If you answer a cube, determine the edge length of the cube.

FQ 13. Determine whether the graph of

$$x^2 + y^2 + z^2 - 4x + 2y + 4z + 9 = 0$$

is the graph of a point, a graph of a sphere or neither.

FQ 14. Determine whether the graph of

$$x^2 + y^2 + z^2 - 2x + 4y - 4z + 7 = 0$$

is the graph of a point, a graph of a sphere or neither.

FQ 15. Determine whether the graph of

$$x^2 + y^2 + z^2 - 2x + 4y - 4z + 10 = 0$$

is the graph of a point, a graph of a sphere or neither.

16. Sketch the graph of $z = \sin y$ using the xyz-coordinate system.

17. Sketch the graph of $x^2 + y^2 = 4$ using the xyz-coordinate system.

18. (a) Describe the set of points that are equidistant from $(1, 2, 3)$ and $(5, 4, 5)$.

(b) How is this set of points related to the line passing through $(1, 2, 3)$ and $(5, 4, 5)$.

(c) Determine at least one point in this set of points.

Answers for §1

1. 5

2. $(3, 4, 1)$

3. $5\sqrt{2}$

4. $\sqrt{41}$

5. $\sqrt{34}$

6. 5

7. (a) 4
   (b) 5

8. The line is parallel to the x-axis, and $y = 4$ and $z = -2$.

9. $(-1, 3, 2)$ and $(-5, 1, 2)$

10. The center is $(1/2, -1, -2)$, and the radius is $\sqrt{34}/2$.

    Comment. After completing the square and rewriting, the equation becomes

    $$4(x - (1/2))^2 + 4(y + 1)^2 + 4(z + 2)^2 = 34.$$

    Dividing both sides by 4, we get the answer.

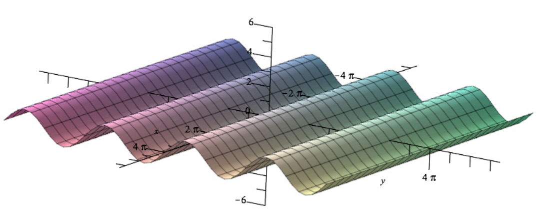
11. The center is $(6, 3, -2)$.

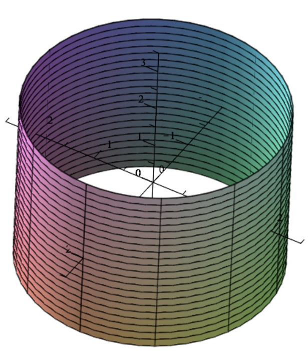
12. The points form a sphere with center $(2, 0, -3)$ and radius 1.

13. The graph is a graph of a point, specifically the point $(2, -1, -2)$.

14. The graph is a graph of a sphere (with center $(1, -2, 2)$ and radius $\sqrt{2}$).

15. This is a graph of neither. This graph is empty (consists of no points).

16. 

17. 

18. (a) The points form a plane.
    (b) The plane is perpendicular to the line through $(1,2,3)$ and $(5,4,5)$.
    (c) The midpoint $(3,3,4)$ of the line segment joining $(1,2,3)$ and $(5,4,5)$ is on the plane.

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§2. Homework Set 2: Introduction to Vectors

FT 1. If $P=(-1,4,2)$ and $Q=(2,-2,6)$, then what is the vector $\overrightarrow{PQ}$?

FT 2. (a) What is the length of the vector $\vec{u}=\langle 2,-6,3\rangle$? Simplify your answer.
(b) Find a unit vector that has the same direction as $\vec{u}$ in part (a).

FF 3. Throughout this problem, $P=(1,3,-1)$ and $Q=(-1,0,5)$. Simplify all answers.
(a) Calculate the distance from P to Q.
(b) Determine the midpoint of the line segment with endpoints P and Q.
(c) What is the vector going from P to Q? Express your answer in the form $\langle a,b,c\rangle$.
(d) What is the length (or magnitude) of the vector $\overrightarrow{PQ}$ (the vector in part (c))?

FF 4. Let $P=(1,0,-2)$ and $Q=(-2,2,4)$.
(a) Calculate the vector $\overrightarrow{PQ}$.
(b) Calculate $2\overrightarrow{PQ}-\langle -6,1,9\rangle$.
(c) Calculate the length of $\overrightarrow{PQ}$ (the vector in part (a)).

FQ 5. Express the vector with initial point $(1,-1,2)$ and terminal point $(3,2,0)$ in the form $\langle a,b,c\rangle$.

FQ 6. Calculate the vector $\overrightarrow{P_1P_2}$ where $P_1=(1,5,-6)$ and $P_2=(4,0,0)$.

FQ 7. What is the vector that has initial point $(-3,-1,1)$ and terminal point $(3,2,-1)$? Write your answer in the form $\langle a,b,c\rangle$.

FQ 8. (a) What is the length of the vector $\langle 6,3,-2\rangle$? Simplify your answer.
(b) What is the unit vector going in the direction of $\langle 6,3,-2\rangle$?

FQ 9. Find a unit vector that is in the opposite direction of the vector $6\,\vec{i}-4\,\vec{j}+2\,\vec{k}$.

FQ 10. Find the initial point of the vector $\vec{v}=\langle -3,1,2\rangle$ if the terminal point is $(5,0,-1)$.

FT 11. What is the initial point of the vector $\langle 3,-3,-4\rangle$ if the terminal point is $(-2,4,1)$?

FQ 12. Calculate the vector that has the opposite direction as $\langle 16, -2, -8\rangle$ but has length 3.

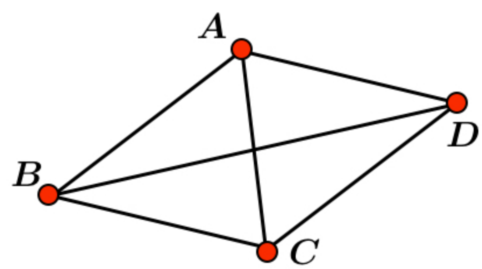
FQ 13. Find a vector of length 3 that goes in the opposite direction as the vector $\langle 8, -1, -4\rangle$.

FQ 14. Find a vector that has the same direction as $\langle -2, 4, 2\rangle$ but has length 6.

FQ 15. Calculate the vector of length 2 going in the opposite direction as $\langle 6, -2, 3\rangle$. Simplify your answer.

FQ 16. Calculate the vector of length 3 going in the opposite direction as $\langle 4, -4, 7\rangle$. Simplify your answer.

FQ 17. In the picture below, $\square ABCD$ is a parallelogram so that the opposite sides are parallel and equal in length. Express the sum $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{CD}$ as a vector $\overrightarrow{PQ}$ where each of P and Q are points from $\{A, B, C, D\}$. (There may be more than one correct answer; you only need to determine one.)



FQ 18. Using the picture in the problem above, express the sum $\overrightarrow{CA} + \overrightarrow{AD} + \overrightarrow{DB}$ as a vector $\overrightarrow{PQ}$ where each of P and Q are points from $\{A, B, C, D\}$.

FT 19. (a) If $P = (1, 0, -1)$ and $Q = (3, -2, 1)$, then what is the vector $\overrightarrow{PQ}$?

(b) Calculate $\langle 3, -4, 1\rangle - 2\langle 1, 2, -1\rangle$.

(c) If $R = (\sqrt{34}, \pi - 17, 789)$, then what is the value of $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP}$? (Hint: You may want to give a little thought to the question, before doing any arithmetic.)

Answers for §2

1. $\overrightarrow{PQ} = \langle 3, -6, 4\rangle$

2. (a) 7

   (b) $\langle 2/7, -6/7, 3/7\rangle$

3. (a) 7

   (b) $(0, 3/2, 2)$

(c) $\langle -2, -3, 6 \rangle$

(d) 7

4. (a) $\langle -3, 2, 6 \rangle$

   (b) $\langle 0, 3, 3 \rangle$

   (c) 7

5. $\langle 2, 3, -2 \rangle$

6. $\langle 3, -5, 6 \rangle$

7. $\langle 6, 3, -2 \rangle$

8. (a) 7

   (b) $\langle 6/7, 3/7, -2/7 \rangle$

9. $\langle -3/\sqrt{14}, 2/\sqrt{14}, -1/\sqrt{14} \rangle$

10. $(8, -1, -3)$

11. $(-5, 7, 5)$

12. $\langle -8/3, 1/3, 4/3 \rangle$

13. $\langle -8/3, 1/3, 4/3 \rangle$

14. $\langle -\sqrt{6}, 2\sqrt{6}, \sqrt{6} \rangle$

15. $\langle -12/7, 4/7, -6/7 \rangle$

16. $\langle -4/3, 4/3, -7/3 \rangle$

17. $\overrightarrow{AB}$ (or $\overrightarrow{DC}$)

18. $\overrightarrow{CB}$ (or $\overrightarrow{DA}$)

19. (a) $\langle 2, -2, 2 \rangle$

    (b) $\langle 1, -8, 3 \rangle$

    (c) $\langle 0, 0, 0 \rangle$

§3. Homework Set 3: Dot Product and Projections

FT 1. Calculate the dot product of the vectors $\overrightarrow{u} = \langle -2, 3, 2 \rangle$ and $\overrightarrow{v} = \langle 4, -1, 1 \rangle$.

FT 2. Calculate the angle θ in $[0, \pi]$ between the vectors $\overrightarrow{u} = \langle 1, -2, 2 \rangle$ and $\overrightarrow{v} = \langle 1, 0, 1 \rangle$. Simplify your answer so that it does not involve inverse trigonometric functions.

FT 3. Are the vectors $\langle 1, 2, -1 \rangle$ and $\langle 2, 1, 3 \rangle$ perpendicular, parallel, or neither?

FF 4. Let $\overrightarrow{a} = \langle 2, 6, 9 \rangle$ and $\overrightarrow{b} = \langle -2, 9, -6 \rangle$. Calculate each of the following, and simplify your answers.

(a) $\|\overrightarrow{a}\|$ (the magnitude of $\overrightarrow{a}$)

(b) $\overrightarrow{a} \bullet \overrightarrow{b}$

FQ 5. In each case, determine whether $\overrightarrow{u}$ and $\overrightarrow{v}$ make an acute angle, an obtuse angle, or are orthogonal.

(a) $\overrightarrow{u} = 6\,\overrightarrow{i} + \overrightarrow{j} + 3\,\overrightarrow{k}$ and $\overrightarrow{v} = 4\,\overrightarrow{i} - 6\,\overrightarrow{k}$

(b) $\overrightarrow{u} = \langle 4, 1, 6 \rangle$ and $\overrightarrow{v} = \langle -3, 0, 2 \rangle$

FF 6. Let $\overrightarrow{u} = \langle 1, -2, 2 \rangle$ and $\overrightarrow{v} = \langle -1, 0, -1 \rangle$. Calculate the smallest angle θ between these two vectors. Simplify your answer so that it does not involve any inverse trigonometric functions.

FQ 7. Calculate the smallest angle between the vectors $\overrightarrow{u} = \langle 1, -1, 4 \rangle$ and $\overrightarrow{v} = \langle 1, 2, -2 \rangle$. Simplify your answer.

FQ 8. What is the measure of the smallest angle between the vectors

$$\overrightarrow{u} = \langle 2, 2, 0 \rangle \quad \text{and} \quad \overrightarrow{v} = \langle -1, 4, -1 \rangle\,?$$

Simplify your answer.

FT 9. What is the smallest angle between the vectors $\langle 2, -3, 1 \rangle$ and $\langle 1, 2, -3 \rangle$? Simplify your answer.

FQ 10. Calculate the smallest angle between the vectors $\overrightarrow{u} = \overrightarrow{j} - \overrightarrow{k}$ and $\overrightarrow{v} = \overrightarrow{i} - \overrightarrow{j} - 4\,\overrightarrow{k}$. Simplify your answer (using either radians or degrees). Show your work.

FT 11. Calculate the angle between the vectors $\langle 1, 0, 1 \rangle$ and $\langle 4, 1, -1 \rangle$.

FQ 12. Determine the angle $\angle ABC$ if $A = (3, -3, -2)$, $B = (2, -1, 1)$ and $C = (4, 2, 2)$. Simplify your answer.

FT 13. Let $A = (-2, 1, -2)$, $B = (-1, 0, 2)$ and $C = (-2, 0, 3)$. What is the value of $\angle ABC$ in degrees? Simplify your answer.

FQ 14. Determine which of the following are true and which are false. Note that "true" means that the statement is always true and "false" means that there is at least one situation where the statement is "false".

(a) The dot product of two vectors is a vector.

(b) If θ is the smallest angle between the vectors $\vec{u}$ and $\vec{v}$, then $\vec{u} \bullet \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$.

(c) If $\vec{a} \bullet \vec{b} = \vec{a} \bullet \vec{c}$ and $\vec{a} \neq \vec{0}$, then $\vec{b} = \vec{c}$.

(d) If $\vec{a} \bullet \vec{b} = \vec{0}$ and $\vec{a} \neq \vec{0}$, then $\vec{b} = \vec{0}$.

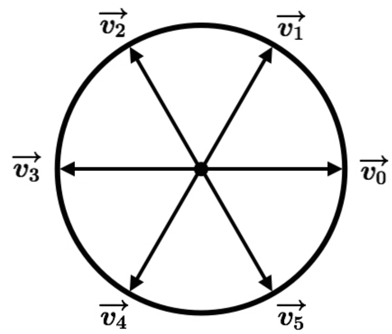
FQ 15. Find the dot product $\vec{u} \bullet \vec{v}$ if $\|\vec{u}\| = 3$, $\|\vec{v}\| = 4$, and the smallest angle between $\vec{u}$ and $\vec{v}$ is $2\pi/3$ (in radians). Simplify your answer.

FQ 16. If $\vec{u}$ and $\vec{v}$ are vectors with $\vec{u}$ having length 2, $\vec{v}$ having length 3 and the angle between $\vec{u}$ and $\vec{v}$ being $\pi/3$, then what is the value of $\vec{u} \bullet \vec{v}$?

FQ 17. The figure to the right shows six vectors that are equally spaced around a circle of radius 5. Find each of the dot products indicated below.

a) $\vec{v_0} \bullet \vec{v_1}$

b) $\vec{v_0} \bullet \vec{v_3}$



FQ 18. Let $A = (-1, 0, 2)$, $B = (1, -4, 6)$ and $C = (2, -1, -5)$. Find the length of the projection of the vector $\overrightarrow{AB}$ onto the vector $\overrightarrow{AC}$. Simplify your answer.

FT 19. Calculate the projection $\text{proj}_{\vec{v}} \vec{u}$ where $\vec{u} = \langle 1, -2, 2 \rangle$ and $\vec{v} = \langle 1, 0, 1 \rangle$.

FQ 20. (a) Let $A = (0, 0, 0)$, $B = (2, 2, -1)$ and $C = (2, -2, 3)$ (these are points in space). Calculate the distance from point B to the line through A and C using vectors.

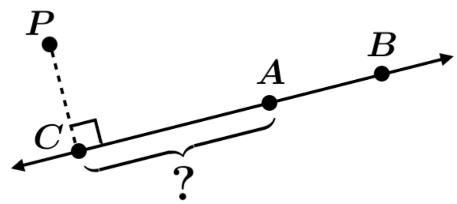
(b) Using the information from part (a), find the area of triangle $\triangle ABC$.

FQ 21. Find the distance from the point $P = (3, 6, 0)$ to the line through the points $A = (1, 0, -1)$ and $B = (3, 1, 1)$.

FQ 22. Find the distance from the point $P = (3, 6, -2)$ to the line which passes through the points $A = (-2, 2, 1)$ and $B = (-1, 0, -1)$. Simplify your answer.

FQ 23. Find the distance from the point $P = (-1, 3, 1)$ and the line passing through the points $A = (-1, 3, -2)$ and $B = (0, 1, 0)$. Simplify your answer.

FQ 24. A perpendicular is drawn from the point $P = (2, -2, 1)$ to the line ℓ through the points $A = (-2, -1, -2)$ and $B = (-4, 0, -4)$, intersecting ℓ at the point C as shown. What is the length of the line segment $\overline{AC}$?



(Not drawn perfectly, but it's pretty good.)

FT 25. A perpendicular is drawn from the point $P = (-2, 2, 5)$ to the line ℓ through the points $A = (-1, 2, 1)$ and $B = (0, 0, -1)$, intersecting ℓ at the point C (see the figure in the previous problem). What is the length of the line segment $\overline{AC}$?

Answers for §3

1. -9
2. $\pi/4$
3. neither
4. (a) 11
   (b) -4
5. (a) acute
   (b) orthogonal
6. $3\pi/4$
7. $3\pi/4$
8. $\pi/3$
9. $2\pi/3$
10. $\pi/3$
11. $\pi/3$
12. $2\pi/3$
13. $2\pi/3$

14. (a) False
    (b) True
    (c) False
    (d) False
15. -6
16. 3
17. (a) $25/2$
    (b) -25
18. $18/\sqrt{59}$
19. $\langle 3/2, 0, 3/2 \rangle$
20. (a) $12/\sqrt{17}$
    (b) 6
21. 5
22. 7
23. $\sqrt{5}$
24. 5
25. 3

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§4. Homework Set 4: Cross Product

FF 1. Let $\overrightarrow{u} = \langle 2, -1, -2 \rangle$ and $\overrightarrow{v} = \langle 3, 1, -1 \rangle$. Calculate each of the following.

(a) $\overrightarrow{u} - 2\overrightarrow{v}$

(b) the dot product of $\overrightarrow{u}$ and $\overrightarrow{v}$

(c) $\overrightarrow{u} \times \overrightarrow{v}$

(d) the length (or magnitude) of $\overrightarrow{u}$

FF 2. Let $\overrightarrow{u} = \langle 2, 1, -2 \rangle$ and $\overrightarrow{v} = \langle 4, -1, -1 \rangle$. Calculate the following.

(a) $2\overrightarrow{u} - \overrightarrow{v}$

(b) $\|\overrightarrow{u}\|$ (the magnitude of $\overrightarrow{u}$)

(c) the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$ (simplify so no inverse trig functions are in your answer)

(d) $\overrightarrow{u} \times \overrightarrow{v}$

FF 3. Throughout this problem, $P = (6, 4, 0)$, $Q = (4, 1, -1)$, and $R = (7, 2, -3)$.

(a) Calculate the vector $\overrightarrow{QP}$.

(b) Calculate the magnitude of the vector $\overrightarrow{QP}$.

(c) Calculate $\angle PQR$ and simplify your answer (do not use inverse trigonometric functions).

(d) Calculate the area of $\triangle PQR$.

(e) Determine a point S such that P, Q, R, and S are the four vertices of a parallelogram. Justify your answer. (There is more than one correct answer.)

FT 4. Let $P = (1, 0, -1)$, $Q = (5, -2, -1)$ and $R = (3, 0, -2)$.

(a) Calculate the vector $\overrightarrow{PQ}$.

(b) Calculate $\cos\theta$ where θ is the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

(c) There are two unit vectors that are each perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. What are they?

(d) What is the area of triangle $\triangle PQR$?

FT 5. Let $P = (1, 4, 1)$, $Q = (2, 3, 5)$ and $R = (1, 3, 6)$.

(a) What is the distance from point Q to the z-axis?

(b) Calculate $\overrightarrow{QP}$.

(c) Calculate $\angle PQR$.

(d) What is the area of $\triangle PQR$?

FT 6. Let $P = (3, -1, 3)$, $Q = (2, 1, 3)$ and $R = (1, -1, 1)$.

(a) Calculate $\overrightarrow{QR}$.

(b) What is the distance from point Q to point R?

(c) What is the area of $\triangle PQR$?

(d) What is the height of $\triangle PQR$ from P to the base $\overline{QR}$?

FT 7. Let $P = (1, 1, -1)$, $Q = (2, -1, 1)$ and $R = (3, 0, -3)$.

(a) Calculate the distance from Q to the z-axis.

(b) Calculate $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

(c) Find a vector of length 2 having the same direction as $\overrightarrow{PQ}$.

(d) What is the area of $\triangle PQR$?

(e) Calculate $\|\mathrm{proj}_{\overrightarrow{PR}}\overrightarrow{PQ}\|$.

(f) What is the height of $\triangle PQR$ from Q to the base $\overline{PR}$?

FQ 8. For each part below, the symbol $\bullet$ represents a dot product and the symbol $\times$ represents a cross product. Determine whether each of the following expressions is "meaningful".

(a) $(\vec{a} \bullet \vec{b}) \times (\vec{c} \bullet \vec{d})$

(b) $(\vec{a} \times \vec{b}) \bullet (\vec{c} \times \vec{d})$

(c) $\vec{a} \bullet (\vec{b} \times \vec{c})$

(d) $\vec{a} \times (\vec{b} \bullet \vec{c})$

(e) $\vec{a} \times (\vec{b} \times \vec{c})$

FQ 9. Calculate $\vec{u} \bullet (\vec{v} \times \vec{w})$, where

$$\vec{u} = \langle 1, -7, -2 \rangle, \quad \vec{v} = \langle 7, -2, -4 \rangle \quad \text{and} \quad \vec{w} = \langle 5, 1, 0 \rangle.$$

FQ 10. Calculate a unit vector perpendicular to $\langle 3, -1, 4 \rangle$ and $\langle 3, 2, -2 \rangle$. Simplify your answer.

FQ 11. Find two unit vectors orthogonal to both $\langle 2, 1, 0 \rangle$ and $\langle 2, 0, -1 \rangle$.

FQ 12. Calculate the area of the triangle with vertices

$$P = (1, 5, -2), \quad Q = (0, 0, 0) \quad \text{and} \quad R = (3, 5, 1).$$

FT 13. What is the area of the triangle with vertices $(1, -1, 2)$, $(2, -2, 4)$ and $(3, -2, 3)$?

FF 14. Calculate the area of the triangle with vertices $(1, 3, 1)$, $(1, 5, 4)$, and $(2, 3, -1)$.

FF 15. Find the area of the triangle with vertices $(-1, 2)$, $(3, 4)$ and $(2, -3)$.

FT 16. (a) What is the area of the triangle with vertices $P = (-1, 0, 2)$, $Q = (0, 0, 3)$, and $R = (7, 4, 3)$? Simplify your answer.

(b) If $S = (0, 0, 1)$, then what is the volume of the parallelepiped having as edges the three segments $\overline{PQ}$, $\overline{PR}$, and $\overline{PS}$?

FT 17. Let $A = (3, -5, -5)$, $B = (1, 0, -1)$ and $C = (-1, 2, 0)$. What is the height of triangle $\triangle ABC$ from the vertex A to the base $\overline{BC}$? Simplify your answer.

FT 18. Three vertices of a parallelogram are $P = (2, 1, 5)$, $Q = (-3, 2, 5)$, and $R = (-3, 1, 6)$. Calculate the area of the parallelogram.

FQ 19. Calculate the area of the parallelogram that has

$$\vec{u} = \langle 1, 0, 1\rangle \quad \text{and} \quad \vec{v} = \langle 1, 2, 2\rangle$$

as adjacent sides.

FT 20. (a) The three points $A = (1, 2, 2)$, $B = (3, 1, 1)$ and $C = (2, -1, 1)$ are three vertices of a parallelogram. The parallelogram has the line segment $\overline{AB}$ as an edge. There is more than one possibility for what the fourth vertex might be. What is one possibility for the fourth vertex?

(b) Calculate the area of the parallelogram with the four vertices in part (a). (The parallelogram is the one obtained with the three vertices given in part (a) and your answer from part (a).)

(c) What is the height of the parallelogram in part (b) using the edge $\overline{AB}$, joining the vertices $(1, 2, 2)$ and $(3, 1, 1)$, as the base of the parallelogram?

FF 21. Let $\vec{a} =< 7, 4, 0 >$, $\vec{b} =< -6, -4, 1 >$ and $\vec{c} = \vec{a} \times \vec{b}$. Viewing these three vectors as emanating from the origin, they form three edges of a parallelepiped. Calculate the volume of the parallelepiped. Simplify your answer.

FT 22. Calculate the volume of the parallepiped which has the vectors $\langle 2, 3, 1\rangle$, $\langle 0, 1, -1\rangle$ and $\langle 0, 0, -3\rangle$ as adjacent edges.

FT 23. (a) What is the volume of the parallelepiped which has the vectors $\langle 5, 0, 0\rangle$, $\langle -4, -3, 0\rangle$ and $\langle 4, 3, 1\rangle$ as adjacent edges?

(b) What is the height of the parallelepiped in part (a) where the base is determined by edges formed from $\langle -4, -3, 0\rangle$ and $\langle 4, 3, 1\rangle$?

FT 24. (a) What is the volume of the parallelepiped with adjacent edges determined by the vectors $\langle -1, 0, -2\rangle$, $\langle 2, 3, 1\rangle$ and $\langle 1, 3, 0\rangle$? Simplify your answer.

(b) What's the height of the parallelepiped in part (a) if the base is determined by the edges $\langle 2, 3, 1\rangle$ and $\langle 1, 3, 0\rangle$?

FQ 25. Consider the parallelepiped with adjacent edges

$$\vec{u} = 3\,\vec{i} + 2\vec{j} + \vec{k}, \quad \vec{v} = \vec{i} + \vec{j} + 2\,\vec{k}$$
$$\text{and} \quad \vec{w} = \vec{i} + 3\vec{j} + 3\,\vec{k}.$$

Find the angle between $\vec{u}$ and the plane containing the face determined by $\vec{v}$ and $\vec{w}$.

FF 26. Find the volume of the pyramid with vertices $P = (-1, 1, -1)$, $Q = (2, 1, 1)$, $R = (-1, 2, 0)$ and $S = (-1, 1, 0)$.

FF 27. Find the volume of the pyramid with vertices $P = (0, 1, 1)$, $Q = (2, 2, 3)$, $R = (0, 4, 3)$, and $S = (0, 1, 3)$.

FT 28. (a) Let $\overrightarrow{u} = \langle x_1, x_2, x_3\rangle$, $\overrightarrow{v} = \langle y_1, y_2, y_3\rangle$, and $\overrightarrow{w} = \langle z_1, z_2, z_3\rangle$. What's the value of $\overrightarrow{w} \bullet (\overrightarrow{u} + \overrightarrow{v})$ in terms of the components of the vectors?

(b) Let $\overrightarrow{u} = \langle x_1, x_2, x_3\rangle$, $\overrightarrow{v} = \langle y_1, y_2, y_3\rangle$, and $\overrightarrow{w} = \langle z_1, z_2, z_3\rangle$. What's the value of $\overrightarrow{w} \bullet \overrightarrow{u} + \overrightarrow{w} \bullet \overrightarrow{v}$ in terms of the components of the vectors?

(c) Explain briefly what (a) and (b) have to do with the equation

$$(\overrightarrow{u} + \overrightarrow{v}) \bullet (\overrightarrow{u} + \overrightarrow{v}) = \overrightarrow{u} \bullet \overrightarrow{u} + 2\overrightarrow{u} \bullet \overrightarrow{v} + \overrightarrow{v} \bullet \overrightarrow{v}.$$

(d) If $\|\overrightarrow{u}\| = 4$, $\|\overrightarrow{v}\| = 3$, and $\|\overrightarrow{u} + \overrightarrow{v}\| = 6$, then what's the value of $\cos\theta$ where θ is the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$?

Answers for §4

1. (a) $\langle -4, 3, 0\rangle$
   (b) 7
   (c) $\langle 3, -4, 5\rangle$
   (d) 3

2. (a) $\langle 0, 3, -3\rangle$
   (b) 3
   (c) $\pi/4$
   (d) $\langle -3, -6, -6\rangle$

3. (a) $\langle 2, 3, 1\rangle$
   (b) $\sqrt{14}$
   (c) $\pi/3$
   (d) $7\sqrt{3}/2$
   (e) $(3, 3, 2)$ or $(5, -1, -4)$ or $(9, 5, -2)$

4. (a) $\langle 4, -2, 0\rangle$
   (b) $4/5$
   (c) $\pm(1/3)\langle 1, 2, 2\rangle$
   (d) 3

5. (a) $\sqrt{13}$
   (b) $\langle -1, 1, -4\rangle$

(c) $2\pi/3$
(d) $3\sqrt{3}/2$

6. (a) $\langle -1, -2, -2\rangle$
   (b) 3
   (c) 3
   (d) 2

7. (a) $\sqrt{5}$
   (b) $\overrightarrow{PQ} = \langle 1, -2, 2\rangle$ and $\overrightarrow{PR} = \langle 2, -1, -2\rangle$
   (c) $(2/3)\langle 1, -2, 2\rangle$
   (d) $9/2$
   (e) 0
   (f) 3

8. (a) No
   (b) Yes
   (c) Yes
   (d) No
   (e) Yes

9. 110

10. $(1/7)\langle -2, 6, 3\rangle$

11. $(\pm 1/3)\langle -1, 2, -2\rangle$

12. $\sqrt{374}/2$

13. $\sqrt{11}/2$

14. $\sqrt{29}/2$

15. 13

16. (a) $9/2$
    (b) 8

17. 3

18. $\sqrt{51}$

19. 3

20. (a) $(4,-2,0)$ or $(0,0,2)$
    (b) $\sqrt{30}$
    (c) $\sqrt{5}$

21. 81

22. 6

23. (a) 15
    (b) 3

24. (a) 3
    (b) $3/\sqrt{19}$

25. $\cos^{-1}(-9/14) - \pi/2 = \pi/2 - \cos^{-1}(9/14)$

26. 1/2

27. 2

28. (a) $z_1(x_1+y_1) + z_2(x_2+y_2) + z_3(x_3+y_3)$
    (b) $z_1x_1 + z_2x_2 + z_3x_3 + z_1y_1 + z_2y_2 + z_3y_3$
    (c) From (a) and (b), we see that

$$\overrightarrow{w} \cdot (\overrightarrow{u} + \overrightarrow{v}) = \overrightarrow{w} \cdot \overrightarrow{u} + \overrightarrow{w} \cdot \overrightarrow{v}. \tag{$*$}$$

Taking $\overrightarrow{w} = \overrightarrow{u} + \overrightarrow{v}$ in $(*)$, we obtain

$$(\overrightarrow{u} + \overrightarrow{v}) \cdot (\overrightarrow{u} + \overrightarrow{v}) = \overrightarrow{w} \cdot (\overrightarrow{u} + \overrightarrow{v}) = \overrightarrow{w} \cdot \overrightarrow{u} + \overrightarrow{w} \cdot \overrightarrow{v}. \tag{$**$}$$

From $(*)$ with some relabelling, we also see that

$$\overrightarrow{w} \cdot \overrightarrow{u} = (\overrightarrow{u} + \overrightarrow{v}) \cdot \overrightarrow{u} = \overrightarrow{u} \cdot (\overrightarrow{u} + \overrightarrow{v}) = \overrightarrow{u} \cdot \overrightarrow{u} + \overrightarrow{u} \cdot \overrightarrow{v}$$

and

$$\overrightarrow{w} \cdot \overrightarrow{v} = (\overrightarrow{u} + \overrightarrow{v}) \cdot \overrightarrow{v} = \overrightarrow{v} \cdot (\overrightarrow{u} + \overrightarrow{v}) = \overrightarrow{v} \cdot \overrightarrow{u} + \overrightarrow{v} \cdot \overrightarrow{v}.$$

Since $\overrightarrow{v} \cdot \overrightarrow{u} = \overrightarrow{u} \cdot \overrightarrow{v}$, we deduce from $(**)$ that

$$(\overrightarrow{u} + \overrightarrow{v}) \cdot (\overrightarrow{u} + \overrightarrow{v}) = \overrightarrow{w} \cdot \overrightarrow{u} + \overrightarrow{w} \cdot \overrightarrow{v}$$

$$= \overrightarrow{u} \bullet \overrightarrow{u} + 2\overrightarrow{u} \bullet \overrightarrow{v} + \overrightarrow{v} \bullet \overrightarrow{v},$$

which is what we set out to explain.

(d) 11/24

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§5. Homework Set 5: Lines

FF 1. Let $P = (11, 3, 8)$ and $Q = (2, 9, 6)$ for each part of this problem.

(a) Calculate the vector $\overrightarrow{PQ}$.

(b) Calculate the distance from P to Q.

(c) Write the parametric equations for the line passing through P and Q.

FT 2. Find parametric equations for the line parallel to the line given by

$$x = -1 + 2t, \quad y = 2 - t, \quad z = 1 + t$$

and passing through the point $(-2, 2, 3)$.

FQ 3. Find parametric equations of the line through the point $(-2, 0, 5)$ that is parallel to the line given by $x = 1 + 2t$, $y = 4 - t$, and $z = 6 + 2t$.

FQ 4. Find parametric equations for the line that passes through $(1, -1, 0)$ and $(0, 2, -1)$.

FT 5. Let $A = (1, 0, 1)$, $B = (-1, 0, 2)$ and $C = (2, 1, 3)$. Let ℓ be the line passing through A and B. Find parametric equations for the line passing through C that is parallel to ℓ.

FQ 6. Find the point of intersection of the lines ℓ_1 and ℓ_2 given by the parametric equations below.

$$\ell_1 : \begin{cases} x &= 2 + t \\ y &= 3 - t \\ z &= 1 - 2t \end{cases} \qquad \ell_2 : \begin{cases} x &= 1 - t \\ y &= 2 + 5t \\ z &= 2 + 4t \end{cases}$$

FT 7. Find the point where the lines below intersect.

$$\ell_1 : \begin{cases} x &= 3t \\ y &= t \\ z &= -1 + t \end{cases} \qquad \ell_2 : \begin{cases} x &= t \\ y &= 2 - t \\ z &= -2 + t \end{cases}$$

FQ 8. The lines L_1 and L_2 given by the parametric equations below intersect at a point. Find the point of intersection. Justify your answer with appropriate work.

$$L_1: \quad x = 2 - t, \quad y = 1 + t, \quad z = 2 + t$$
$$L_2: \quad x = 2t, \qquad y = 4 - t, \quad z = 3 - 3t$$

FT 9. The lines given by the parametric equations $x = 1+4t$, $y = -4t$ and $z = 2+6t$ and $x = 6+t$, $y = 1 + t$ and $z = -1 - 2t$ intersect. What is their point of intersection?

FT 10. (a) Show that the lines

$$\ell_1: \begin{cases} x &= 1 - 6t \\ y &= 1 + 3t \\ z &= 1 \end{cases} \qquad \ell_2: \begin{cases} x &= 4 + t \\ y &= 12 + 2t \\ z &= -4 - t \end{cases}$$

intersect and find their point of intersection.

(b) Explain why the lines above are perpendicular.

FT 11. (a) Show that the lines L_1 and L_2 below intersect and find their point of intersection.

$$L_1: \quad x = 2 + 2t, \quad y = 3 - 2t, \quad z = 6t$$
$$L_2: \quad x = 1 + 3t, \quad y = 3t, \quad z = 7 - 6t$$

(b) Explain why the lines L_1 and L_2 above are *not* perpendicular.

FT 12. (a) Show that the two lines ℓ and ℓ' below intersect and find their point of intersection.

$$\ell: \begin{cases} x &= 4 + 2t \\ y &= -t \\ z &= 4 + 3t \end{cases} \qquad \ell': \begin{cases} x &= -4 + t \\ y &= 11 - 4t \\ z &= -15 + 5t \end{cases}$$

(b) What is the measure of the smallest angle between the two lines in part (a)? Simplify your answer so that it does *not* use an inverse trigonometric function.

FQ 13. (a) The lines ℓ_1 and ℓ_2, given by the parametric equations below, intersect at a point. Calculate the point.

$$\ell_1: \quad x = 1 + 2t, \quad y = -2 - 2t, \quad z = 4t$$
$$\ell_2: \quad x = 1 + t, \qquad y = -1 + 2t, \quad z = -1 - t$$

(b) Calculate the smallest angle between the lines ℓ_1 and ℓ_2 in part (a). Simplify your answer.

FT 14. The two lines ℓ_1 and ℓ_2 below intersect. What is the measure of the smallest angle between ℓ_1 and ℓ_2? Simplify your answer so that it does NOT use an inverse trigonometric function.

$$\ell_1 : \begin{cases} x = 2014 - 2t \\ y = 2009 - t \\ z = 2010 + t \end{cases} \qquad \ell_2 : \begin{cases} x = 2013 + t \\ y = 2014 + t \\ z = 2016 \end{cases}$$

FT 15. Let ℓ be the line given by the parametric equations $x = 3t$, $y = 1 - 2t$ and $z = 1 + 2t$. Let ℓ' be the line given by the parametric equations $x = 2 - t$, $y = 2 + t$ and $z = -t$. The lines ℓ and ℓ' intersect at a point P. Write parametric equations for the line ℓ'' perpendicular to both ℓ and ℓ' and passing through P.

FQ 16. Find the distance from the point $P = (3, 6, 0)$ to the line through the points $A = (1, 0, -1)$ and $B = (3, 1, 1)$.

FQ 17. Calculate the distance between the lines ℓ_1 and ℓ_2 given by the parametric equations below.

$$\ell_1 : \begin{cases} x = 1 + t \\ y = -1 - 2t \\ z = -1 - t \end{cases} \qquad \ell_2 : \begin{cases} x = 2 - t \\ y = 3 + t \\ z = -3 + 2t \end{cases}$$

FQ 18. The two lines given by the parametric equations

$$x = 1 - 2t, \quad y = 2 + 2t, \quad z = 1 + t$$

and

$$x = 1 + 2t, \quad y = -2t, \quad z = 2 - t$$

are parallel. Calculate the distance between the lines.

FQ 19. Lines ℓ_1 and ℓ_2 are given by the following parametric equations.

$$\ell_1 : \begin{cases} x = -t \\ y = 1 - t \\ z = t \end{cases} \qquad \ell_2 : \begin{cases} x = s \\ y = 1 - s \\ z = 1 - s \end{cases}$$

(a) Explain why the lines do NOT intersect. Use complete English sentences and be precise.

(b) Explain why the lines are NOT parallel. Use complete English sentences and be precise.

(c) Using vectors, calculate the distance between ℓ_1 and ℓ_2.

FT 20. Calculate the distance between the two lines ℓ_1 and ℓ_2 below (i.e., find the minimum distance from a point on ℓ_1 to a point on ℓ_2).

$$\ell_1 : \begin{cases} x = 2 + t \\ y = 0 \\ z = -1 + t \end{cases} \qquad \ell_2 : \begin{cases} x = 4 \\ y = 2t \\ z = 1 + t \end{cases}$$

FT 21. Let A be the point $(1,-2,-2)$. Let ℓ be the line given by the parametric equations $x = -1-2t$, $y = 2+t$ and $z = 3t$. Calculate the point C on ℓ closest to the point A.

FT 22. Calculate the point P that lies on the line segment from $A = (2,1,5)$ to $B = (8,-2,14)$ with the distance from A to P equal to twice the distance from P to B.

FT 23. Let $P = (2,1,4)$ and $Q = (-1,7,7)$. Determine with justification which of the following points are on the line "segment" $\overline{PQ}$ (the line segment with endpoints P and Q). Give an explanation for your answer.

$$(0,4,6), \quad (0,5,6), \quad (1,3,5), \quad (1,4,6), \quad (-4,13,10)$$

FT 24. Let L be the line given by

$$L: \quad x = 6+2t,\ y = 3+t,\ z = -6-2t.$$

(a) Explain why the point $(2,1,-2)$ is on line L.

(b) Find a point P also on line L that is a distance 2010 units from the point $(2,1,-2)$. (There are 2 correct answers. You only need to find one such P.)

FT 25. What is the distance between the two lines ℓ_1 and ℓ_2 given below.

$$\ell_1: \begin{cases} x &= 2t \\ y &= 1-t \\ z &= 2+t \end{cases} \qquad \ell_2: \begin{cases} x &= 3 \\ y &= 1-t \\ z &= -t \end{cases}$$

FT 26. Calculate the distance between the lines ℓ_1 and ℓ_2 given by the parametric equations below. Simplify your answer.

$$\ell_1: \begin{cases} x = 1+2t \\ y = -1-2t \\ z = -1-t \end{cases} \qquad \ell_2: \begin{cases} x = 2+t \\ y = 3-4t \\ z = -3+t \end{cases}$$

FF 27. Let ℓ_1 be the line with parametric equations $x = 1-4t$, $y = 1-t$, and $z = -1+t$, and let ℓ_2 be the line with parametric equations $x = 3$, $y = 1-2t$, and $z = 1+t$. Then ℓ_1 and ℓ_2 are two lines in space which do not intersect. Calculate the distance between these lines. Simplify your answer.

FT 28. What point on the line ℓ given by the parameterization $x = 2-4t$, $y = 1+5t$ and $z = -3t$ is closest to the sphere $x^2+y^2+z^2-2x+6y+4z+5 = 0$?

FF 29. Let $P = (2,1,-2)$, and let ℓ be the line given by the parametric equations $x = 1+t$, $y = 4+2t$ and $z = -t$. There are two lines ℓ_1 and ℓ_2 which each pass through P and intersect ℓ at a 60° angle. Each of these lines ℓ_1 and ℓ_2 intersects ℓ. Determine the two points of intersection, that is where ℓ_1 and ℓ intersect and where ℓ_2 and ℓ intersect.

FQ 30. (a) Find parametric equations for the line through the point $(5, 4, -7)$ that is parallel to the line given by $x = t$, $y = -3 - t$ and $z = 4 + 2t$.

(b) Find parametric equations for the line through the point $(5, 4, -7)$ that is perpendicular to the line $x = t$, $y = -3 - t$ and $z = 4 + 2t$ and intersects this line.

FF 31. Let ℓ_1 and ℓ_2 be two lines given by parametric equations as follows:

$$\ell_1 : \begin{cases} x = 1 + t \\ y = -2 + t \\ z = -t \end{cases} \qquad \ell_2 : \begin{cases} x = 2 - t \\ y = 3 + 2t \\ z = 1 - t \end{cases}$$

(a) Explain why the lines ℓ_1 and ℓ_2 are skew lines. In other words, explain why they do not intersect and why they are not parallel.

(b) Since ℓ_1 and ℓ_2 are skew lines, there must be exactly one line ℓ_3 that intersects both ℓ_1 and ℓ_2 and is perpendicular to both ℓ_1 and ℓ_2. Find parametric equations for the line ℓ_3.

Answers for §5

1. (a) $\langle -9, 6, -2 \rangle$

   (b) 11

   (c) $x = 11 - 9t,\ y = 3 + 6t,\ z = 8 - 2t$

2. $x = -2 + 2t,\ y = 2 - t,\ z = 3 + t$

3. $x = -2 + 2t,\ y = -t,\ z = 5 + 2t$

4. $x = 1 - t,\ y = -1 + 3t,\ z = -t$

5. $x = 2 - 2t,\ y = 1,\ z = 3 + t$

6. $(1/2, 9/2, 4)$

7. $(3/2, 1/2, -1/2)$

8. $(-2, 5, 6)$

9. $(3, -2, 5)$

10. (a) $(-1, 2, 1)$

(b) Vectors parallel to ℓ_1 and ℓ_2 are $\langle -6, 3, 0\rangle$ and $\langle 1, 2, -1\rangle$, respectively. Since

$$\langle -6, 3, 0\rangle \bullet \langle 1, 2, -1\rangle = 0,$$

the lines ℓ_1 and ℓ_2 perpendicular.

11. (a) $(3, 2, 3)$

    (b) Vectors parallel to L_1 and L_2 are $\langle 2, -2, 6\rangle$ and $\langle 3, 3, -6\rangle$, respectively. Since

    $$\langle 2, -2, 6\rangle \bullet \langle 3, 3, -6\rangle = -36 \neq 0,$$

    the lines L_1 and L_2 are not perpendicular.

12. (a) $(-2, 3, -5)$

    (b) $\pi/6$

13. (a) $(2/3, -5/3, -2/3)$

    (b) $\pi/3$

14. $\pi/6$

15. $x = 9,\ y = -5 + t,\ z = 7 + t$

16. 5

17. $5/\sqrt{11}$

18. 2

19. (a) If there were a point (x_0, y_0, z_0) on both lines, then the equations for ℓ_1 would imply $x_0 + z_0 = 0$ and the equations for ℓ_2 would imply $x_0 + z_0 = 1$. Since $0 \neq 1$, this is impossible, so the lines do not intersect.

    (b) A vector parallel to ℓ_1 is $\langle -1, -1, 1\rangle$, and a vector parallel to ℓ_2 is $\langle 1, -1, -1\rangle$. For the lines to be parallel, we would need $\langle -1, -1, 1\rangle = k\langle 1, -1, -1\rangle = \langle k, -k, -k\rangle$ for some number k. Equating first components gives $-1 = k$ and equating second components gives $-1 = -k$. Since it is impossible for $k = -1$ and $k = 1$, we see that the lines ℓ_1 and ℓ_2 are not parallel.

    (c) $1/\sqrt{2}$

20. 0

21. $(1, 1, -3)$

22. $(6, -1, 11)$

23. $(0,5,6)$ and $(1,3,5)$

24. (a) Plugging in $t=-2$ in the equations for L, one obtains the point $(2,1,-2)$. So $(2,1,-2)$ is on L.

    (b) $(1342,671,-1342)$ or $(-1338,-669,1338)$

25. $5/\sqrt{3}$

26. $2/3$

27. 2

28. $(14/5,0,3/5)$

29. $(-1/3,4/3,4/3)$ or $(4/3,14/3,-1/3)$

30. (a) $x=5+t,\ y=4-t,\ z=-7+2t$

    (b) $x=-4+3t,\ y=1+t,\ z=-4-t$

31. (a) If (x_0,y_0,z_0) were on both ℓ_1 and ℓ_2, then $x_0+2y_0+3z_0=-3$ (since (x_0,y_0,z_0) satisfies the equations for ℓ_1) and $x_0+2y_0+3z_0=11$ (since (x_0,y_0,z_0) satisfies the equations for ℓ_2). Since $-3\neq 11$, this does not happen, and ℓ_1 and ℓ_2 do not intersect.

    The vectors $\langle 1,1,-1\rangle$ and $\langle -1,2,-1\rangle$ are parallel to ℓ_1 and ℓ_2, respectively. If the lines were parallel, there would be a number k such that $\langle 1,1,-1\rangle = k\langle -1,2,-1\rangle$. Looking at first components in this equation, we would have $k=-1$. Looking at second components in the equation, we would have $k=1/2$. Since $-1\neq 1/2$, this does not happen, and the lines are not parallel.

    (b) $x=2+t,\ y=-1+2t,\ z=-1+3t$

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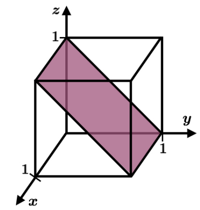
§6. Homework Set 6: Planes (and More Lines)

FF 1. Determine parametric equations for the line perpendicular to the plane $x+2y-3z=4$ and passing through the point $(0,1,-1)$.

FF 2. Determine parametric equations for the line perpendicular to the plane $x-y+2z=4$ and passing through the point $(1,0,-1)$.

FT 3. The line given by the paremetric equations $x=1+t$, $y=2$, and $z=2+t$ is on a plane $\mathcal{P}$. The plane $y+z=4$ is perpendicular to plane $\mathcal{P}$. Find the equation for plane $\mathcal{P}$.

FQ 4. Find an equation for the plane shown in the figure to the right.



FT 5. What is the equation of the plane that is perpendicular to the plane $2x - y + z = 3$ and passes through the points $(2, 4, 0)$ and $(3, 3, -1)$?

FQ 6. Find the equation of the plane containing the line $x = 1 + 2t, y = -1 + 3t, z = 4 + t$ and the point $(2, -1, 4)$.

FQ 7. (a) If $P = (3, 1, 2)$ and the line ℓ is given by the parametric equations $x = 1 - t$, $y = -2t$ and $z = 2 + t$, then what is an equation for the plane passing through the point P and perpendicular to the line ℓ?

(b) With P and ℓ as in part (a), what is an equation for the plane passing through the point P and containing the line ℓ?

FT 8. The 2 planes $x - y - z = 1$ and $x + y = 2$ intersect in a line. Find parametric equations for the line.

FQ 9. Calculate the distance between the two planes $x + 2y - 2z = 1$ and $x + 2y - 2z = 2$.

FQ 10. Find an equation for the plane through $(1, 2, -1)$ that is perpendicular to the line of intersection of the planes

$$2x + y + z = 2 \quad \text{and} \quad x + 2y + z = 3.$$

FQ 11. Find parametric equations of the line through the point $(1, 0, 2)$ that is parallel to the planes $x + y - z = 2$ and $x - y + 2z = -1$.

FQ 12. Find parametric equations for the line that passes through the point $(1, 1, 2)$ and is parallel to the line of intersection of the two planes $x + 2y + z = -1$ and $x - y + 2z = 3$.

FT 13. (a) Let ℓ_1 and ℓ_2 be the lines given below. Justify with work that these two lines intersect at one point and find their point of intersection.

$$\ell_1 : \begin{cases} x = 1 - 2t \\ y = 1 - t \\ z = 1 + t \end{cases} \qquad \ell_2 : \begin{cases} x = 1 + t \\ y = 2 - t \\ z = t \end{cases}$$

(b) Find an equation for the plane that contains both the lines ℓ_1 and ℓ_2 from part (a).

FT 14. The points that are equidistant from $(-1, 3, 2)$ and $(3, 1, 2)$ form a plane. Calculate an equation for this plane.

FQ 15. Let ℓ_1 and ℓ_2 be the lines below. The line ℓ_1 is on a plane $\mathcal{P}$ that is parallel to the line ℓ_2.

Find an equation for the plane $\mathcal{P}$.

$$\ell_1 : \begin{cases} x = 1 + t \\ y = -1 - 2t \\ z = -1 - t \end{cases} \qquad\qquad \ell_2 : \begin{cases} x = 2 - t \\ y = 3 + t \\ z = -3 + 2t \end{cases}$$

FF 16. (a) Explain why the line given by the parametric equations $x = 2 - t$, $y = t$, and $z = 2 + t$ is on the plane $3x - y + 4z = 14$.

(b) The line in part (a) is on a plane $\mathcal{P}$ that is perpendicular to the plane in part (a). What is the equation of plane $\mathcal{P}$?

FT 17. The points $(-1, 0, 2)$ and $(3, -1, 1)$ are on a plane $\mathcal{P}$. The plane $\mathcal{P}$ is also perpendicular to the plane $2x - y + z = 2010$. Find an equation for the plane $\mathcal{P}$.

FT 18. Find an equation for the plane $\mathcal{P}$ that is perpendicular to the plane $2x - y + z = 0$ and passes through the points $(0, 1, -1)$ and $(2, 1, 1)$.

FT 19. Let $\mathcal{P}_1$ be the plane whose equation is given by $2x - 3y + z = 4$, and let $\mathcal{P}_2$ be the plane whose equation is given by $x - y + z = 2$. Let A be the point $(2, 1, 2)$. Find parametric equations for the line ℓ which passes through the point A and is parallel to both plane $\mathcal{P}_1$ and plane $\mathcal{P}_2$.

FT 20. (a) Explain why the two planes

$$x + 2y - z = 1 \qquad \text{and} \qquad -x - 2y + z = 1$$

are parallel. Be sure to include in your explanation a reason why the two equations are not two different equations for the same plane.

(b) Using vectors, calculate the distance between the two planes given in part (a)?

FF 21. Calculate the distance between the planes $x + 4y + 8z = -3$ and $x + 4y + 8z = 15$. Simplify your answer.

FQ 22. (a) Find the shortest distance from the point $A = (4, -7, -1)$ to the plane $\mathcal{P}$ defined by $2x - 2y + z = 6$. Simplify your answer.

(b) At what point does the line ℓ below intersect the plane $\mathcal{P}$ given in part (a)?

$$\ell : \begin{cases} x = 3 + 3t \\ y = 1 + 3t \\ z = -2 - 2t \end{cases}$$

FF 23. Let $\mathcal{P}$ be the plane $3x + y - 4z = 7$, let ℓ_1 be the line given by $x = 1 + 2t$, $y = 1 - 2t$, and $z = 3 + t$, and let ℓ_2 be the line given by $x = -1 + t$, $y = 2 + t$, and $z = -2 + t$.

(a) Explain why ℓ_1 does not intersect the plane $\mathcal{P}$.

(b) Does ℓ_2 intersect the plane $\mathcal{P}$? Justify your answer.

(c) Calculate the minimum distance from ℓ_1 to ℓ_2.

FF 24. Given the point $A = (1, -1, 2)$ and the plane $\mathcal{P}$ given by $x + 2y - 3z = 0$, calculate the point B on the plane $\mathcal{P}$ that is nearest to A.

FF 25. Let $\mathcal{P}$ be the plane given by the equation $x - y + z = 2$. The point $Q = (-1, 2, 1)$ is not on the plane $\mathcal{P}$. Find an equation for a plane which passes through the point Q and is *perpendicular* to the plane $\mathcal{P}$.

FT 26. Let $\mathcal{P}$ be the plane $x + y - z = 2$. Find the equation of a plane perpendicular to $\mathcal{P}$ and passing through the points $(1, 4, -3)$ and $(1, 5, -2)$.

FF 27. Let A be the point $(1, -2, -2)$. Let $\mathcal{P}$ be the plane given by $-2x + y + 3z = 4$. Calculate the point B on $\mathcal{P}$ that is closest to A.

FT 28. The line given by the paremetric equations $x = 1 + t$, $y = 2$, and $z = 2 + t$ is on a plane $\mathcal{P}$. The plane $y + z = 4$ is perpendicular to plane $\mathcal{P}$. Find an equation for plane $\mathcal{P}$.

FT 29. Calculate an equation for the plane that is perpendicular to the plane $x + y + 2z = 2014$ and contains the line ℓ given by the parameterization $x = 2014 - 3t$, $y = 2014 + t$ and $z = 2014 + 2t$.

FT 30. Let $\mathcal{P}_1$ be the plane given by $2x - 3y + z = 2014$, and let $\mathcal{P}_2$ be the plane given by $x - y - z = 2014$. Let A be the point $(1, 1, 0)$. Find parametric equations for the line ℓ passing through the point A and parallel to both plane $\mathcal{P}_1$ and plane $\mathcal{P}_2$.

FT 31. Calculate an equation for the plane that is perpendicular to the plane $2x - 3y + z = 3$ and contains the line ℓ given by the parameterization $x = -1 + t$, $y = 2t$ and $z = 2 - t$.

FT 32. Find an equation for the plane that is parallel to the two lines ℓ_1 and ℓ_2 below and that passes through the point $(1, 2, 0)$.

$$\ell_1 : \begin{cases} x &= 2t \\ y &= 1 - t \\ z &= 2 + t \end{cases} \qquad \ell_2 : \begin{cases} x &= 3 \\ y &= 1 - t \\ z &= -t \end{cases}$$

FT 33. Find an equation of a plane through point P and parallel to line ℓ given below.

$$P = (2, -1, 1) \qquad \text{and} \qquad \ell : \begin{cases} x &= t \\ y &= -t \\ z &= 2 - t \end{cases}$$

FT 34. (a) Show that the lines ℓ_1 and ℓ_2 below intersect and find their point of intersection.

$$\ell_1: \ x = 2 - t, \ \ y = \quad 2t, \ \ z = -2 + 2t$$
$$\ell_2: \ x = 1 + 2t, \ \ y = 3 - t, \ \ z = \quad 3 + 5t$$

(b) What is an equation for the plane $\mathcal{P}$ parallel to ℓ_1 and ℓ_2 in the previous problem and passing through the origin $(0, 0, 0)$?

(c) Justify (if you didn't already) that the lines ℓ_1 and ℓ_2 are not on the plane $\mathcal{P}$.

FT 35. (a) Let ℓ be the line given by

$$\ell : \begin{cases} x &= 1 - t \\ y &= -1 + 2t \\ z &= 1 - 2t \end{cases}.$$

Let $\mathcal{P}$ be the plane with equation $-2x + y + 2z = -6$. Explain why ℓ is parallel to $\mathcal{P}$. (In other words, explain why there is no point on both ℓ and $\mathcal{P}$.)

(b) For ℓ and $\mathcal{P}$ as in part (a), find the distance from ℓ to $\mathcal{P}$.

FT 36. (a) Let $\mathcal{P}$ be the plane $x - y + 2z = 3$. Find the parametric equations of the line ℓ that is perpendicular to $\mathcal{P}$ and passes through the point $(0, 0, 1)$.

(b) Find the point of intersection of the plane $\mathcal{P}$ and the line ℓ given in part (a).

(c) Using part (b), calculate the distance from the point $(0, 0, 1)$ and the plane $\mathcal{P}$.

FT 37. The plane $\mathcal{P}_1$ has the equation $x + 2z = 4$, and the plane $\mathcal{P}_2$ has the equation $x - y = 2$. They intersect in a line ℓ. Calculate the equation of the *plane* containing the line ℓ and the point $(1, 1, 1)$.

FT 38. (a) Determine the parametric equations for a line that is perpendicular to the plane $2x + y - z = 4$ and that passes through the point $(-1, 2, 0)$.

(b) Determine the parametric equations for a line that is *parallel* to the plane $2x + y - z = 4$ and that passes through the point $(-1, 2, 0)$. (There is more than one correct answer.)

FT 39. Find parametric equations in the variable t for a line segment of length 1 that contains the point $(1, 1, 1)$ on the plane $6x + 2y - 3z = 5$ as an endpoint and is perpendicular to this plane. Your answer should indicate the appropriate values for the variable t in the form $a \leq t \leq b$ where a and b are numbers.

FT 40. For each of the parts below, the plane $\mathcal{P}$ has the equation $6x - 2y - 3z = 1$. The point $A = (1, 1, 1)$ is on the plane $\mathcal{P}$.

(a) Find parametric equations for the line ℓ which is perpendicular to the plane $\mathcal{P}$ and passes through the point A.

(b) Find a point B on the line ℓ in part (a) such that the distance from A to B is 14. There are two such points B. You only need to give me one of them.

(c) What's an equation for a plane $\mathcal{Q}$ parallel to $\mathcal{P}$ and a distance 14 from $\mathcal{P}$ (that is, the minimal distance from each point on $\mathcal{Q}$ to the plane $\mathcal{P}$ is 14)? There are two such planes $\mathcal{Q}$. You only need to find one of them.

FF 41. For each part below, the lines ℓ_1 and ℓ_2 are given by

$$\ell_1 : \begin{cases} x = 2 + 2t \\ y = -1 + 4t \\ z = -2t \end{cases} \qquad \ell_2 : \begin{cases} x = 3 + 2t \\ y = -1 - 2t \\ z = 1 + 4t \end{cases}$$

(a) Show that the lines intersect, and find the point A of intersection.

(b) Lines ℓ_1 and ℓ_2 both are in a plane $\mathcal{P}$. Find an equation for the plane $\mathcal{P}$.

(c) Find parametric equations for the line ℓ_3 that bisects the smallest angle formed by ℓ_1 and ℓ_2. In other words, ℓ_3 lies in $\mathcal{P}$, ℓ_3 passes through the point A, and the smallest angle between the lines ℓ_1 and ℓ_3 is the same as one-half the smallest angle between the lines ℓ_1 and ℓ_2.

FF 42. For each part below, the lines ℓ_1 and ℓ_2 are given by

$$\ell_1 : \begin{cases} x = t - 2 \\ y = -3t \\ z = 3t - 1 \end{cases} \qquad \ell_2 : \begin{cases} x = 1 + t \\ y = 3t + 1 \\ z = -3t - 2 \end{cases}$$

(a) Show that the lines intersect, and find the point A of intersection.

(b) Line ℓ_1 and ℓ_2 both are in a plane $\mathcal{P}$. Find an equation for the plane $\mathcal{P}$.

(c) Find parametric equations for the line ℓ_3 that bisects the smallest angle formed by ℓ_1 and ℓ_2. In other words, ℓ_3 lies in $\mathcal{P}$, ℓ_3 passes through the point A, and the smallest angle between the lines ℓ_1 and ℓ_3 is the same as one-half the smallest angle between the lines ℓ_1 and ℓ_2.

FF 43. Let $\mathcal{P}$ be the plane containing the points $A = (0, 0, 0)$, $B = (5, 2, 4)$, and $C = (-4, -4, -2)$. Let $\mathcal{T}$ be the triangle $\triangle ABC$ in the plane $\mathcal{P}$ including the edges $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$, and all points enclosed by them in $\mathcal{P}$.

(a) What is an equation for the plane $\mathcal{P}$?

(b) Explain why $(2, 0, 2)$ is in triangle $\mathcal{T}$ but $(2, 0, 1)$ is not.

(c) Does the line ℓ given by the parametric equations

$$x = t + 2, \qquad y = t + 1, \qquad \text{and} \qquad z = -3t - 2$$

intersect $\mathcal{T}$? Justify your answer.

FF 44. Let $\mathcal{P}$ be the plane $x - 2y - z = 3$. The points $Q = (3, -2, 4)$ and $R = (4, -1, 3)$ are both on the plane $\mathcal{P}$. There are two planes $\mathcal{P}'$ and $\mathcal{P}''$ each intersecting plane $\mathcal{P}$ at a 60° angle and each passing through both Q and R. The purpose of this problem is to find equations for the planes $\mathcal{P}'$ and $\mathcal{P}''$.

(a) If the plane $ax + by + cz = d$ is either $\mathcal{P}'$ and $\mathcal{P}''$ so that it intersects $\mathcal{P}$ at a 60° angle (that is the smallest angle between these two planes is 60°), explain why

$$a - 2b - c = \pm\frac{\sqrt{6}}{2}\sqrt{a^2 + b^2 + c^2}.$$

(b) The points (x, y, z) that satisfy $ax + by + cz = d$ are the same as the points (x, y, z) that satisfy $atx + bty + ctz = dt$ for any non-zero number t (in other words, if we multiply both sides of an equation for a plane by a non-zero constant, then the new equation still describes the same plane). This implies that we can alter the coefficients of x, y, and z describing the plane $ax + by + cz = d$ so that the sum of the squares of the coefficients is 6 (or whatever else we want). Suppose then that $a^2 + b^2 + c^2 = 6$. Show why if the plane $ax + by + cz = d$ is as in part (a), then

$$a - 2b - c = \pm 3.$$

(c) The vector $\overrightarrow{QR}$ is related to the plane $ax + by + cz = d$ of part (a). Explain why

$$a + b - c = 0$$

using the vector $\overrightarrow{QR}$. (In addition to the mathematics, write an English sentence that explains the connection between $\overrightarrow{QR}$ and the plane $ax + by + cz = d$.)

(d) Use the equations $a - 2b - c = \pm 3$, $a + b - c = 0$, and $a^2 + b^2 + c^2 = 6$ from parts (b) and (c) and the equation $a^2 + b^2 + c^2 = 6$ to determine two possibilities for (a, b, c) (where again the plane $ax + by + cz = d$ is as in part (a)).

(e) What are equations for the planes $\mathcal{P}'$ and $\mathcal{P}''$ in the discussion before part (a)?

Answers for §6

1. $x = t,\ y = 1 + 2t,\ z = -1 - 3t$
2. $x = 1 + t,\ y = -t,\ z = -1 + 2t$
3. $x + y - z = 1$
4. $y + z = 1$
5. $2x + 3y - z = 16$
6. $y - 3z = -13$

7. (a) $x + 2y - z = 3$
   (b) $x - 2y - 3z = -5$

8. $x = 1 + t,\ y = 1 - t,\ z = -1 + 2t$

9. $1/3$

10. $x + y - 3z = 6$

11. $x = 1 + t,\ y = -3t,\ z = 2 - 2t$

12. $5x - y - 3z = -2$

13. (a) $(5/3, 4/3, 2/3)$
    (b) $y + z = 2$

14. $2x - y = 0$

15. $3x + y + z = 1$

16. (a) The line is on the plane since $3(2 - t) - t + 4(2 + t) = 14$ for every value of t.
    (b) $5x + 7y - 2z = 6$

17. $x + 3y + z = 1$

18. $x + y - z = 2$

19. $x = 2 + 2t,\ y = 1 + t,\ z = 2 - t$

20. (a) The second plane equation can be written as $x+2y-z = -1$. We see then that $\langle 1, 2, -1\rangle$ is perpendicular to both planes. Also, $(1, 0, 0)$ is on the first plane but not the second. Hence, the planes are different and parallel.
    (b) $2/\sqrt{6}$

21. 2

22. (a) 5
    (b) $(-3, -5, 2)$

23. (a) The line ℓ_1 does not intersect $\mathcal{P}$ since $3(1 + 2t) + (1 - 2t) - 4(3 + t) = -8 \neq 7$.
    (b) Yes. In fact, ℓ_2 is on $\mathcal{P}$. (The justification is not provided here.)
    (c) Note $\langle 2, -2, 1\rangle \neq k\langle 1, 1, 1\rangle$ for any k, so the lines are not parallel. The minimum distance

is the distance from any point on ℓ_1 to $\mathcal{P}$. The answer is $15/\sqrt{26}$.

24. $(3/2, 0, 1/2)$

25. There are infinitely many such planes. One is $x + y = 1$.

26. $2x - y + z = -5$

27. $(-1, -1, 1)$

28. $x + y - z = 1$

29. $2y - z = 2014$

30. $4x + 3y + z = 7$

31. $x + 3y + 7z = 13$

32. $x + y - z = 3$

33. There are infinitely many such planes. One is $x + y = 1$.

34. (a) $(1/3, 10/3, 4/3)$

    (b) $4x + 3y - z = 0$

    (c) The point $(2, 0, -2)$ is on ℓ_1. The point $(2, 0, -2)$ is not on $4x + 3y - z = 0$ since $8 + 0 - (-2) = 10 \neq 0$. Since the lines intersect and have directional vectors parallel to $\mathcal{P}$, this is enough to ensure the lines ℓ_1 and ℓ_2 are both not on $\mathcal{P}$.

35. (a) There is no point on both ℓ and $\mathcal{P}$ since $-2(1-t) + (-1+2t) + 2(1-2t) = -1 \neq -6$.

    (b) $5/3$

36. (a) $x = t,\ y = -t,\ z = 1 + 2t$

    (b) $(1/6, -1/6, 4/3)$

    (c) $1/\sqrt{6}$

37. $x + y + 4z = 6$

38. (a) $x = -1 + 2t,\ y = 2 + t,\ z = -t$

    (b) One answer is $x = -1,\ y = 2 + t,\ z = t$.

39. $x = 1 + 6t,\ y = 1 + 2t,\ z = 1 - 3t$ where $0 \leq t \leq 1/7$

40. (a) $x = 1 + 6t,\ y = 1 - 2t,\ z = 1 - 3t$

(b) $(13, -3, -5)$ or $(-11, 5, 7)$

(c) $6x - 2y - 3z = 99$ or $6x - 2y - 3z = -97$

41. (a) $(7/3, -1/3, -1/3)$

(b) $x - y - z = 3$

(c) $x = 7/3,\ y = (-1/3) + t,\ z = (-1/3) - t$

42. (a) $(-2/3, -4, 3)$

(b) $y + z = -1$

(c) $x = t - 2,\ y = -4\ z = 3$

43. (a) $2x - y - 2z = 0$

(b) The point $(2, 0, 2)$ is on $\mathcal{P}$ since $2 \cdot 2 - 0 - 2 \cdot 2 = 0$. The line segment $\overline{BC}$ has the parametric representation $x = -4 + 9t$, $y = -4 + 6t$, and $z = -2 + 6t$ where $0 \le t \le 1$. Taking $t = 2/3$, we see that $(2, 0, 2)$ is on $\overline{BC}$. Therefore, $(2, 0, 2)$ is on side $\overline{BC}$ of $\mathcal{T}$ and hence in $\mathcal{T}$. The point $(2, 0, 1)$ is not on $\mathcal{P}$ since $2 \cdot 2 - 0 - 2 \cdot 1 = 2 \neq 0$, so $(2, 0, 1)$ is not a point in $\mathcal{T}$.

(c) Yes. To see this, we first determine where ℓ intersects $\mathcal{P}$. Since

$$2(t + 2) - (t + 1) - 2(-3t - 2) = 7t + 7 = 0$$

if and only if $t = -1$, the line ℓ intersects $\mathcal{P}$ at the point $(1, 0, 1)$. Let $U = (2, 0, 2)$. From part (b), the point U is on $\overline{BC}$. Therefore, the midpoint of $\overline{AU}$ is in $\mathcal{T}$. Since this midpoint is $(1, 0, 1)$, the line ℓ intersects $\mathcal{T}$ at $(1, 0, 1)$.

41. (a) Let $\mathcal{P}_0$ be the plane $ax + by + cz = d$. The vector $\langle a, b, c\rangle$ is perpendicular to $\mathcal{P}_0$. The vector $\langle 1, -2, -1\rangle$ is perpendicular to $\mathcal{P}$. By the conditions in the problem, the smallest angle θ between these two vectors is either $\pi/3$ or $2\pi/3$. Using the cosine formula for the angle between two vectors, we get

$$\pm\frac{1}{2} = \cos\theta = \frac{\langle 1, -2, -1\rangle \bullet \langle a, b, c\rangle}{\sqrt{6} \cdot \sqrt{a^2 + b^2 + c^2}} = \frac{a - 2b - c}{\sqrt{6} \cdot \sqrt{a^2 + b^2 + c^2}}.$$

Multiplying by $\sqrt{6} \cdot \sqrt{a^2 + b^2 + c^2}$ gives the equation stated in part (a).

(b) Taking $a^2 + b^2 + c^2 = 6$ in part (a), we get

$$a - 2b - c = \pm\frac{\sqrt{6}}{2}\sqrt{a^2 + b^2 + c^2} = \pm\frac{\sqrt{6}}{2}\sqrt{6} = \pm 3.$$

(c) The vector $\overrightarrow{QR} = \langle 1, 1, -1\rangle$ is parallel to $\mathcal{P}_0$. The vector $\langle a, b, c\rangle$ is perpendicular to $\mathcal{P}_0$. Therefore, $\langle 1, 1, -1\rangle \bullet \langle a, b, c\rangle = 0$, which is equivalent to $a + b - c = 0$.

(d) Observe that if (a, b, c) is a solution to the equations $a - 2b - c = \pm 3$, $a + b - c = 0$, and $a^2 + b^2 + c^2 = 6$, then so is $(-a, -b, -c)$. This is because the equation $ax + by + cz = d$ for $\mathcal{P}_0$ can also be written as $-ax - by - cz = -d$. So as to only consider one of these equations for $\mathcal{P}_0$, we restrict to $a - 2b - c = 3$ (since $a - 2b - c = -3$ will just come from replacing (a, b, c) with $(-a, -b, -c)$).

Taking a difference with the equations $a - 2b - c = 3$ and $a + b - c = 0$ gives $b = -1$. Substituting $b = -1$ into $a + b - c = 0$ and $a^2 + b^2 + c^2 = 6$ gives $a = c + 1$ and $a^2 + c^2 = 5$. Hence,

$$5 = a^2 + c^2 = (c + 1)^2 + c^2 = 2c^2 + 2c + 1.$$

Subtracting 5 from both sides and dividing by 2, we deduce that

$$0 = c^2 + c - 2 = (c - 1)(c + 2).$$

Thus, either $c = 1$ and $a = c + 1 = 2$ or $c = -2$ and $a = c + 1 = -1$. Therefore, (a, b, c) is either $(2, -1, 1)$ or $(-1, -1, -2)$.

(e) The point $Q = (3, -2, 4)$ is on $\mathcal{P}_0$. From part (d), we deduce that $\mathcal{P}_0$ is given by one of the two equations

$$2x - y + z = 12 \quad \text{and} \quad x + y + 2z = 9.$$

Since both $\mathcal{P}'$ and $\mathcal{P}''$ exist and we only have two possibilities for what they can be, these two equations are equations for $\mathcal{P}'$ and $\mathcal{P}''$.

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§7. Homework Set 7: Surfaces

1. (a) Sketch the solid that is bounded below by the elliptic paraboloid $z = x^2 + y^2$ and above by the elliptic paraboloid $z = 8 - x^2 - y^2$.

   (b) What is the intersection of these two elliptic paraboloids? Be precise.

2. (a) Sketch the solid that is bounded below by the xy-plane and the top part of the hyperboloid of one sheet $x^2 + y^2 - 2z^2 = 1$ and above by the hemisphere $z = \sqrt{4 - x^2 - y^2}$.

   (b) What is the intersection of the hyperboloid of one sheet and the hemisphere in part (a)? Be precise.

FT 3. The graph of $4x^2 - y^2 + z^2 = 16$ is an hyperboloid of one sheet. There are two points on the graph that are closer (a shorter distance) to the y-axis than the other points on the graph. What are these two points?

FT 4. The graphs for the equations below are similar to the graphs given in "Graph Section I". The orientation and the scaling may be different. For each equation, indicate which graph in the Graph Section best matches it. For example, if the equation is for a hyperbolic paraboloid, then the graph you choose should be a hyperbolic paraboloid. Indicate your choice by writing the corresponding letter from the Graph Section on your paper and circling the letter. Next, read the question in the Graph Section corresponding to the graph you choose. Then answer the question for the graph of the equation in the problem below. Do **NOT** answer the question for the graph in the Graph Section I (since it may be oriented differently than the graph of the equation below).

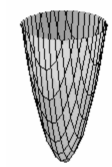
(a) $x^2 - 2y^2 + z^2 + 8 = 0$ (b) $x^2 - 2y^2 - z^2 + 8 = 0$
(c) $x^2 - 2y + z^2 + 8 = 0$ (d) $z^2 + 2x^2 = 3y^2$
(e) $z + 2x^2 = 3y^2$ (f) $2x^2 + z^2 = 3y^2 - 1$
(g) $2x^2 + z^2 = 3y - 1$ (h) $x^2 - y^2 + z^2 + 1 = 0$
(i) $x^2 - y^2 + z + 1 = 0$ (j) $x^2 - y + z^2 + 1 = 0$
(k) $x^2 - y^2 + 3z = 0$ (l) $x^2 + y^2 + 3z = 1$
(m) $x - y^2 - 4z^2 = -4$ (n) $z^2 = x^2 - 2y^2 - 4$
(o) $z^2 = x^2 + 2y^2 - 4$ (p) $z = x^2 + 2y^2 - 4$
(q) $x^2 - y^2 - 4z^2 = 5$ (r) $x^2 - y^2 + 4z^2 = -5$
(s) $x^2 - y^2 + 4z^2 = 0$ (t) $2x^2 - 5y^2 - 4z^2 = 0$
(u) $2x - 5y^2 - 4z^2 = 3$ (v) $2x^2 - 5y^2 - 4z^2 = -3$
(w) $-x^2 - y^2 + 2z^2 = -2$ (x) $-x^2 - y^2 + 2z^2 = 2$
(y) $-x - y^2 + 2z^2 = -2$ (z) $2x^2 - y^2 - 4z^2 = 0$

FT 5. Repeat the previous problem but using "Graph Section II" instead of "Graph Section I".

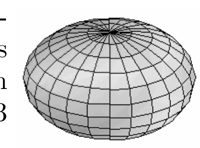
(a) $x^2 + 2y^2 + 3z^2 = 4$ (b) $x^2 + 2y^2 - 3z^2 = 0$
(c) $x + 2y^2 - 3z^2 = 0$ (d) $4x^2 + y^2 + z^2 = 16$
(e) $4x^2 - y^2 + z^2 = 16$ (f) $4x^2 - y^2 + z = 16$
(g) $4x^2 - y + 5z^2 + 1 = 0$ (h) $3x^2 - y^2 - 3z^2 = 0$
(i) $x^2 - 3y^2 - 4z^2 - 2 = 0$ (j) $-x^2 + y^2 - 2z^2 = 0$
(k) $-x^2 - y^2 - 2z^2 + 3 = 0$ (l) $-x^2 + y^2 + 2z^2 + 3 = 0$
(m) $x^2 - 2y^2 = 3z^2$ (n) $x^2 - 2y^2 = 3z^2 - 1$
(o) $1 + 2x^2 - y^2 - 4z^2 = 0$ (p) $1 - 2x^2 - y^2 - 4z^2 = 0$
(q) $x^2 = y^2 + 3z^2$ (r) $x^2 + 2y^2 + z^2 - 9 = 0$
(s) $x^2 - 2y^2 + 3z^2 + 4 = 0$ (t) $x^2 - y - z^2 = 0$
(u) $x^2 - y^2 - 4z^2 = -1$ (v) $x^2 - y^2 - 4z^2 = 0$
(w) $x - y^2 - 4z^2 = 0$

Graph Section I

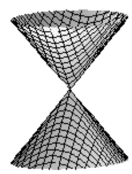
(a) To the right is a graph of an elliptic paraboloid. What point is the vertex in the graph for the problem? Give all 3 of the coordinates.



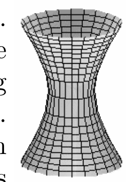
(b) This is a graph of an ellipsoid. What are the 2 points where the graph for the problem intersects the y-axis? Give all 3 coordinates for each point.



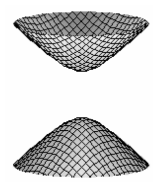
(c) This is an elliptic cone. The xy-plane intersects the graph for the problem in two lines. Write down parametric equations for these two lines.



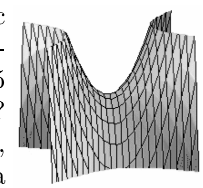
(d) This is a hyperboloid of 1 sheet. Planes that are parallel to a coordinate plane can intersect the corresponding equation in the problem in 2 lines. Find such a plane (there's more than one), and find parametric equations for the 2 lines.



(e) This is a graph of a hyperboloid of 2 sheets. Where are the two vertices of the hyperboloid of 2 sheets in the graph in the problem? For each point, give all 3 coordinates.

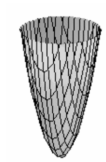


(f) This is a hyperbolic paraboloid. What is the intersection of the plane $y = 2015$ with the graph in the problem? Answer a point, two lines, an ellipse, a parabola, or a hyperbola.

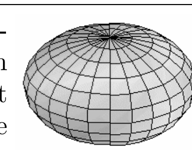


Graph Section II

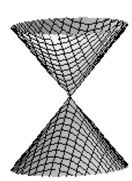
(a) To the right is a graph of an elliptic paraboloid. Which of the three planes $x = 7$, $y = 7$ and $z = 7$ intersects the graph for the problem in an ellipse?



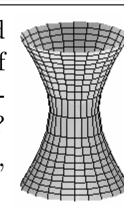
(b) This is a graph of an ellipsoid. What are the 2 points on the graph for the problem that are closest to the origin? Give all 3 coordinates for each point.



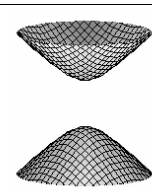
(c) This is an elliptic cone. What is the intersection of the graph for the corresponding equation in the problem with the yz-plane? Answer either a point, lines, an ellipse, a square, a parabola, or a hyperbola.



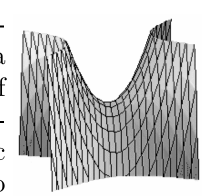
(d) This is a graph of a hyperboloid of 1 sheet. What is the intersection of the graph for the corresponding equation in the problem with the yz-plane? Answer either a point, lines, an ellipse, a square, a parabola, or a hyperbola.



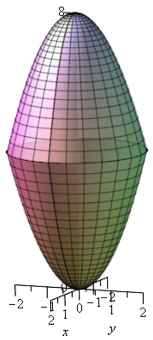
(e) This is a graph of a hyperboloid of 2 sheets. The graph for the corresponding equation in the problem intersects one of the x-axis, y-axis, and z-axis. Which axis does it intersect?



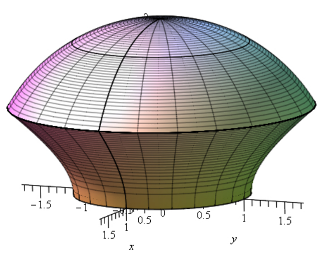
(f) This is a graph of a hyperbolic paraboloid. There is a plane, perpendicular to one of the x, y, or z-axes, that intersects the graph of the hyperbolic paraboloid in the problem in two lines. What is the equation of the plane?



Answers for §7

1. (a) 

   (b) The elliptic paraboloids intersect in a circle of radius 2 centered at $(0, 0, 4)$ and parallel to the xy-plane.

2. (a) 

   (b) The intersection is a circle of radius $\sqrt{3}$ centered at $(0, 0, 1)$ and parallel to the xy-plane.

3. $(\pm 2, 0, 0)$

4. (a) Graph (e); $(0, \pm 2, 0)$

   (b) Graph (d); one answer is the plane $y = 2$ with one line $x = t$, $y = 2$, and $z = t$ and the other line $x = t$, $y = 2$, and $z = -t$

   (c) Graph (a); $(0, 4, 0)$

   (d) Graph (c); $x = t$, $y = \pm\sqrt{2/3}\, t$, and $z = 0$

   (e) Graph (f); a parabola

   (f) Graph (e); $(0, \pm 1/\sqrt{3}, 0)$

   (g) Graph (a); $(0, 1/3, 0)$

   (h) Graph (e); $(0, \pm 1, 0)$

   (i) Graph (f); a parabola

   (j) Graph (a); $(0, 1, 0)$

   (k) Graph (f); a parabola

   (l) Graph (a); $(0, 0, 1/3)$

(m) Graph (a); $(-4, 0, 0)$

(n) Graph (e); $(\pm 2, 0, 0)$

(o) Graph (d); one answer is the plane $x = 2$ with one line $x = 2$, $y = t$, and $z = \sqrt{2}\, t$ and the other line $x = 2$, $y = t$, and $z = -\sqrt{2}\, t$

(p) Graph (a); $(0, 0, -4)$

(q) Graph (e); $(\pm\sqrt{5}, 0, 0)$

(r) Graph (e); $(0, \pm\sqrt{5}, 0)$

(s) Graph (c); $x = t$, $y = \pm t$, and $z = 0$

(t) Graph (c); $x = t$, $y = \pm\sqrt{2/5}\, t$, and $z = 0$

(u) Graph (a); $(3/2, 0, 0)$

(v) Graph (d); one answer is the plane $y = \sqrt{3/5}$ with one line $x = \sqrt{2}\, t$, $y = \sqrt{3/5}$, and $z = t$ and the other line $x = -\sqrt{2}\, t$, $y = \sqrt{3/5}$, and $z = t$

(w) Graph (d); one answer is the plane $x = \sqrt{2}$ with one line $x = \sqrt{2}$, $y = \sqrt{2}\, t$, and $z = t$ and the other line $x = \sqrt{2}$, $y = -\sqrt{2}\, t$, and $z = t$

(x) Graph (e); $(0, 0, \pm 1)$

(y) Graph (f); a parabola

(z) Graph (c); $x = t$, $y = \pm\sqrt{2}\, t$, and $z = 0$

5. (a) Graph (b); $(0, 0, \pm 2/\sqrt{3})$

(b) Graph (c); lines

(c) Graph (f); $x = 0$

(d) Graph (b); $(\pm 2, 0, 0)$

(e) Graph (d); a hyperbola

(f) Graph (f); $z = 16$

(g) Graph (a); $y = 7$

(h) Graph (c); a point

(i) Graph (e); the x-axis

(j) Graph (c); lines

(k) Graph (b); $(0, 0, \pm\sqrt{3/2})$

(l) Graph (e); the x-axis

(m) Graph (c); a point

(n) Graph (d); an ellipse

(o) Graph (d); an ellipse

(p) Graph (b); $(0, 0, \pm 1/2)$

(q) Graph (c); a point

(r) Graph (b); $(0, \pm 3/\sqrt{2}, 0)$

(s) Graph (e); the y-axis

(t) Graph (f); $y = 0$

(u) Graph (d); an ellipse

(v) Graph (c); a point

(w) Graph (a); $x = 7$