MATH 580/780I: TEST 2, FALL 2010

Instructions and Point Values: Put your name at the top of the first page of the blank paper given to you. There are 6 problems below. Work each problem on the paper provided, using a separate page for each problem. Circle your answer (not the work) on the blank page for each part of each problem. Show ALL of your work. Do NOT use a calculator.

There are 100 total points possible on this exam. The point value for each problem appears to the left of each problem.

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18 pts (1) Calculate each of the following. Simplify your answers. Each part of this problem should have a short solution based on a theorem or formula from class.

(a) The value of $\phi(2^2 \cdot 3^2 \cdot 5)$.

(b) The smallest positive integer a such that $9^{12} \equiv a \pmod{13}$.

(c) An integer b between -100 and 100 satisfying $12! \equiv b \pmod{13}$.

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18 pts (2) (a) Define the word "pseudoprime".

(b) Prove that each of the following congruences is valid.

$$2^{40} \equiv 1 \pmod{3}, \quad 2^{40} \equiv 1 \pmod{11}, \quad 2^{40} \equiv 1 \pmod{17}, \quad 2^{40} \equiv 1 \pmod{41}$$

(c) Using part (b), explain why $23001 = 3 \cdot 11 \cdot 17 \cdot 41$ is a pseudoprime. You should be able to handle all the cases (each prime) at once.

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20 pts (3) (a) State Euler's Theorem.

(b) Calculate $\phi(1000)$.

(c) Give a one sentence explanation for why Euler's Theorem does NOT imply

$$2^{\phi(1000)} \equiv 1 \pmod{1000}.$$

(d) What is the remainder when $2^{\phi(1000)}$ is divided by 1000? In other words, determine what $R \in \{0, 1, 2, \dots, 999\}$ satisfies $2^{\phi(1000)} \equiv R \pmod{1000}$.

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12 pts (4) Find the smallest positive integer n satisfying all of the following:

- The number $n + 2$ is divisible by 2.
- The number $n + 20$ is divisible by 20.
- The number $n + 201$ is divisible by 201.
- The number $n + 2010$ is divisible by 2010.

Note that $n = 0$ is an "obvious" integer that satisfies the above four conditions. In the problem, I am asking for you to find the next n with this property. You may want to use that $201 = 3 \cdot 67$ and $2010 = 2 \cdot 3 \cdot 5 \cdot 67$.

12 pts (5) What is the remainder when

$$123456789101112\ldots9899100$$

is divided by 90? (The digits of the number displayed above are 1, 2, 3, ..., 100 written side-by-side.)

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20 pts (6) (a) Find the smallest positive integer solution to the system of congruences below.

$$\begin{aligned} x &\equiv 4 \pmod{8} \\ x &\equiv 8 \pmod{12} \\ x &\equiv 12 \pmod{20} \\ x &\equiv 20 \pmod{42} \end{aligned}$$

Simplify your answer (do the arithmetic). This could be a "little" messy. Deal with it.

(b) Solve the system of congruences above. In other words, describe the complete set of integers x that have the property that each x in the set satisfies ALL of the congruences displayed in part (a).