MATH 580/780I: TEST 1, FALL 2010

Instructions and Point Values: Put your name at the top of the first page of the blank paper given to you. There are 7 problems below. Work each problem on the paper provided, using a separate page for each problem. Show ALL of your work. Do NOT use a calculator.

There are 100 total points possible on this exam. The point value for each problem appears to the left of each problem.

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14 pts (1) Prove that $\log_4 7$ is an irrational number. Use complete English sentences.

Proof. Assume $\log_4 7$ is rational. Then there exist integers a and b with $b > 0$ such that $\log_4 7 = a/b$. Since $\log_4 7 > 0$, we know $a > 0$. From properties of logarithms and $\log_4 7 = a/b$, we deduce $\log_4(7^b) = b\log_4 7 = a$ and, hence, $7^b = 4^a$. Since 4^a is even (because $a > 0$) and 7^b is odd, we have a contradiction. Therefore, we obtain that our assumption is incorrect and $\log_4 7$ is irrational. ■

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14 pts (2) Calculate $\gcd(25979, 2573)$ using the Euclidean algorithm. Show your work and circle the value of the greatest common divisor.

$$\begin{aligned} \mathbf{25979} &= \mathbf{2573 \cdot 10 + 249} \\ \mathbf{2573} &= \mathbf{249 \cdot 10 + 83} \\ \mathbf{249} &= \mathbf{83 \cdot 3 + 0} \end{aligned}$$

The answer is 83.

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14 pts (3) Find integers x and y such that $25979x + 2573y = \gcd(25979, 2573)$. You may want to use your calculations from the previous problem. Circle the values of x and y and be sure to clarify which value is x and which value is y.

Since $10 + \dfrac{1}{10} = \dfrac{101}{10}$, we get $25979(-10) + 2573(101) = 83$.

We can take $x = -10$ and $y = 101$.

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14 pts (4) Observe that $(x, y) = (2, -2)$ is a solution to the equation $18x + 15y = 6$. Describe ALL the pairs of integers (x, y) that satisfy the equation $18x + 15y = 6$. Clarify whether any parameters in your answer are intended to be integers, positive integers, rational numbers, etc. Circle your answer.

Observe that $\gcd(18, 15) = \gcd(2 \cdot 3^2, 3 \cdot 5) = 3$, so

$$\frac{15}{\gcd(18,15)} = \frac{15}{3} = 5 \quad \text{and} \quad \frac{18}{\gcd(18,15)} = \frac{18}{3} = 6.$$

The answer is $x = 2 + 5t$ and $y = -2 - 6t$, where t is an arbitrary integer.

14 pts (5) Observe that $2^4 \equiv 16 \equiv 1 \pmod 5$ and $3^4 \equiv 81 \equiv 1 \pmod 5$. Also, $2010 = 4 \cdot 502 + 2$. Using this information, calculate the remainder when $2^{2010} + 3^{2010}$ is divided by 5. Show the work justifying your answer and circle it.

We have

$$\mathbf{2^4 \equiv 16 \equiv 1 \pmod 5 \implies (2^4)^{502} \equiv 1^{502} \pmod 5 \implies 2^{2008} \equiv 1 \pmod 5}$$
$$\mathbf{\implies 2^{2010} \equiv 2^{2008} \cdot 2^2 \equiv 1 \cdot 2^2 \equiv 4 \pmod 5}$$

and

$$\mathbf{3^4 \equiv 81 \equiv 1 \pmod 5 \implies (3^4)^{502} \equiv 1^{502} \pmod 5 \implies 3^{2008} \equiv 1 \pmod 5}$$
$$\mathbf{\implies 3^{2010} \equiv 3^{2008} \cdot 3^2 \equiv 1 \cdot 3^2 \equiv 9 \equiv 4 \pmod 5.}$$

Hence,

$$\mathbf{2^{2010} + 3^{2010} \equiv 4 + 4 \equiv 8 \equiv 3 \pmod 5.}$$

The remainder is 3.

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14 pts (6) Let a, b and d be integers. Using the definition of what it means for one integer to divide another, prove that if d divides a and d divides b, then d divides $2a - 3b$. The proof should be short, but use complete English sentences and make your explanation clear.

Proof. Suppose d divides a and d divides b. By the definition of what it means for one integer to divide another, we obtain there are integers k and ℓ such that $a = dk$ and $b = d\ell$. Hence,

$$\mathbf{2a - 3b = 2(dk) - 3(d\ell) = 2dk - 3d\ell = d(2k - 3\ell).}$$

Since k and ℓ are integers, the number $2k - 3\ell$ is an integer. Thus, $2a - 3b$ is d times an integer. By the definition of what it means for one integer to divide another, we deduce that d divides $2a - 3b$. ■

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16 pts (7) For each part below, determine if there is an integer x satisfying the given congruence. If so, then find a *positive* integer x satisfying the congruence and circle it. If not, explain why no integer x exists. To possibly help you with this problem, I have indicated the complete prime factorization of numbers occurring in this problem and some Euclidean algorithm computations associated with numbers in this problem. You do not need to check this information and can feel free to use it.

(a) $2010\,x \equiv 2 \pmod{162}$

(b) $2010\,x \equiv 2 \pmod{163}$

"Possible" Useful Information

$2010 = 2 \cdot 3 \cdot 5 \cdot 67$	$2010 = 162 \cdot 12 + 66$	$2010 = 163 \cdot 12 + 54$
$162 = 2 \cdot 3^4$	$162 = 66 \cdot 2 + 30$	$163 = 54 \cdot 3 + 1$
$163 = 163$ (it's a prime)	$66 = 30 \cdot 2 + 6$	$54 = 1 \cdot 54 + 0$
	$30 = 6 \cdot 5 + 0$	

(a) Suppose $2010\,x \equiv 2 \pmod{162}$. Then there is an integer y such that

$$2010\,x + 162\,y = 2.$$

From the given information, we see that both 2010 and 162 are divisible by 3. Thus, 3 divides $2010\,x + 162\,y$. Since 3 does not divide 2, we see that the equation $2010\,x + 162\,y = 2$ is impossible. Therefore, there is no integer x (positive or otherwise) satisfying $2010\,x \equiv 2 \pmod{162}$.

(b) In this part, we want to see if there are integers x and y satisfying

$$2010\,x + 163\,y = 2.$$

From the given information, $\gcd(2010, 163) = 1$. Hence, we can find integers x_0 and y_0 such that

$$2010\,x_0 + 163\,y_0 = 1.$$

Using the given information, we deduce from $12 + (1/3) = 37/3$ that

$$2010\,(-3) + 163\,(37) = 1.$$

Multiplying both sides by 2, we obtain

$$2010\,(-6) + 163\,(74) = 2.$$

Hence, $2010\,(-6) \equiv 2 \pmod{163}$. Since $-6 \equiv 157 \pmod{163}$, we obtain that $2010 \cdot 157 \equiv 2 \pmod{163}$. Therefore, we can take $x = 157$.
