Math 580: Quiz 7

Show ALL Work Name **Solutions**

1. Do the following problem from Test 1 <u>without</u> using the Chinese Remainder Theorem. Note that $111 = 3 \cdot 37$, where both 3 and 37 are primes. Find $x \in \{0, 1, \ldots 110\}$ satisfying
$$2^{222} \equiv x \pmod{111}.$$
In other words, what is the remainder when 2^{222} is divided by 111 ? Justify your answer with appropriate work.

   Answer: $\boxed{64}$

   Solution. Since $\phi(111) = 2 \cdot 36 = 72$, we deduce from Euler's Theorem that $2^{72} \equiv 1 \pmod{111}$. Since $222 = 72 \cdot 3 + 6$ (by dividing 222 by 72), we see that
$$2^{222} \equiv 2^{72 \cdot 3 + 6} \equiv \left(2^{72}\right)^3 2^6 \equiv 2^6 \equiv 64 \pmod{111},$$
implying that the answer is 64. ■

2. You may use the Chinese Remainder Theorem on this problem; it is similar to but different from a problem asked on Test 1. The value of $\phi(1000)$ is 400. What are the last three digits (the three right-most digits) of the number 5^{2002} ? Justify your answer with appropriate work and put all three digits in the correct order as they appear from left to right.

   Answer: $\boxed{625}$

   Solution. This can be done by observing that the last 3 digits of 5^k for $k \geq 3$ alternate between being 125 (for k odd) and 625 (for k even). Since 2002 is even, 5^{2002} ends in the digits 625. But you will need to know how to do a problem like this using the Chinese Remainder Theorem (in other words, don't count on an easy pattern for the powers of a number in general). To use the Chinese Remainder Theorem, combine the information from
$$5^2 \equiv 1 \pmod 8 \implies 5^{2002} \equiv 1 \pmod 8 \qquad \text{and} \qquad 5^{2002} \equiv 0 \pmod{125},$$
where the latter congruence comes from the fact that $125 = 5^3$ divides 5^{2002}. Since the Chinese Remainder Theorem tells us that there is a unique number modulo $8 \times 125 = 1000$ satisfying the congruences $x \equiv 1 \pmod 8$ and $x \equiv 0 \pmod{125}$, this number will be the value of 5^{2002} modulo 1000. Using our method for the Chinese Remainder Theorem, we obtain
$$x \equiv 1 \cdot 125 \cdot 5 + 0 \cdot (\text{something}) \equiv 625 \pmod{1000}.$$
Recall the 5 above comes from computing the inverse of $125 \equiv 5$ modulo 8 (that is, we are using here that $5 \cdot 5 \equiv 1 \pmod 8$). ■