Math 580: Quiz 2

Show ALL Work Name Solution Key

1. Explain why the product $(n-1)(n^2+n-3)$ is divisible by 3 for every integer n. Throughout your explanation, use complete English sentences. If you use the Division Algorithm, make sure to indicate where you use it (by saying, for example, "By the Division Algorithm, we get...").

Explanation: By the Division Algorithm, there are integers q and r such that $n = 3q + r$ and $r \in \{0, 1, 2\}$. We consider such q and r and show that, for each choice of r, we get the desired conclusion that 3 divides $(n-1)(n^2+n-3)$.

If $r = 0$, then $n = 3q$ so that

$$\begin{aligned}(n-1)(n^2+n-3) &= (3q-1)\big((3q)^2 + (3q) - 3\big)\\ &= (3q-1)\big(9q^2+3q-3\big)\\ &= 3\big((3q-1)(3q^2+q-1\big).\end{aligned}$$

Thus, $(n-1)(n^2+n-3)$ is 3 times an integer, and we obtain that 3 divides $(n-1)(n^2+n-3)$ if $r = 0$.

If $r = 1$, then $n = 3q + 1$ so that $n - 1 = 3q$. Hence,

$$(n-1)(n^2+n-3) = (3q)(n^2+n-3) = 3\big(q(n^2+n-3)\big).$$

In this case, we also have $(n-1)(n^2+n-3)$ is 3 times an integer, and we obtain that 3 divides $(n-1)(n^2+n-3)$ if $r = 1$.

If $r = 2$, then $n = 3q + 2$ so that

$$\begin{aligned}(n-1)(n^2+n-3) &= \big((3q+2)-1\big)\big((3q+2)^2+(3q+2)-3\big)\\ &= (3q+1)\big(9q^2+12q+4+3q+2-3\big)\\ &= (3q+1)\big(9q^2+15q+3\big)\\ &= 3\big((3q+1)(3q^2+5q+1\big).\end{aligned}$$

Therefore, in this case, we also have $(n-1)(n^2+n-3)$ is 3 times an integer, and we deduce that 3 divides $(n-1)(n^2+n-3)$ if $r = 2$.

In conclusion, we know $r \in \{0, 1, 2\}$; and, for each such r, we have shown 3 divides $(n-1)(n^2+n-3)$. Thus, for every integer n, we obtain that 3 divides the product $(n-1)(n^2+n-3)$. ■