Math 580: Quiz 1

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1. We showed in class that $\sqrt{2}$ is irrational, and you can use that $\sqrt{2}$ is irrational in this problem without proving it. It is not known whether π^e is irrational, and it is not known whether $\pi+e$ is irrational. Using complete English sentences, prove that at least one of the numbers

$$\pi^e, \quad \pi+e, \quad \text{and} \quad \sqrt{2}\left(\pi^e+\sqrt{2}\,(\pi+e)\right)$$

is irrational. (Hint: Make an appropriate assumption, and note that I am not saying that you need to determine which of these three numbers is irrational.)

Solution 1. Assume each of the three numbers is rational. Since they are positive, we get that there are positive integers a, b, c, d, f and g such that

$$\pi^e = \frac{a}{b}, \quad \pi+e = \frac{c}{d}, \quad \text{and} \quad \sqrt{2}\left(\pi^e+\sqrt{2}\,(\pi+e)\right) = \frac{f}{g}.$$

Then

$$\frac{f}{g} = \sqrt{2}\left(\pi^e+\sqrt{2}\,(\pi+e)\right) = \sqrt{2}\,\pi^e + 2\,(\pi+e) = \sqrt{2}\times\frac{a}{b} + 2\times\frac{c}{d}.$$

Solving for $\sqrt{2}$, we obtain

$$\sqrt{2} = \frac{\dfrac{f}{g}-\dfrac{2c}{d}}{\dfrac{a}{b}} = \frac{\dfrac{df-2cg}{dg}}{\dfrac{a}{b}} = \frac{bdf-2bcg}{adg}.$$

Since $\sqrt{2}$ is irrational and the last expression above is rational, we obtain a contradiction. Hence, one of the three numbers given in the problem must be irrational. $\square$

Solution 2 (similar but without a proof by contradiction). Suppose π^e and $\pi+e$ are rational. To finish the proof, we show that in this case $\sqrt{2}\left(\pi^e+\sqrt{2}\,(\pi+e)\right)$ must be rational. Since π^e and $\pi+e$ are rational and are positive, there are positive integers a, b, c and d such that $\pi^e = a/b$ and $\pi+e = c/d$. Then

$$\sqrt{2}\left(\pi^e+\sqrt{2}\,(\pi+e)\right) = \sqrt{2}\,\pi^e + 2\,(\pi+e) = \sqrt{2}\times\frac{a}{b} + 2\times\frac{c}{d} = \sqrt{2}\times\frac{a}{b} + \frac{2c}{d}.$$

Note that an irrational number times a nonzero rational number is irrational. Also, an irrational number plus a rational number is irrational. Since $\sqrt{2}$ is irrational, we deduce that $\sqrt{2}\times(a/b)$ is irrational and then that $\sqrt{2}\times(a/b)+2c/d$ is irrational. Therefore, we obtain $\sqrt{2}\left(\pi^e+\sqrt{2}\,(\pi+e)\right)$ is irrational, completing the proof. $\square$

Comment. Neither solution is showing which of the three numbers given is irrational. They are just showing that one of them must be irrational. If you could determine which ones are irrational, you would be doing something that no one else in the world, past or present, knows how to do.