Math 580: Quiz 12

Show ALL Work Name Key

1. Finish the proof below that there exist infinitely many primes $\not\equiv 1 \pmod{2010}$. To finish the proof, you should explain the following:

   - Why are all the prime divisors of N, given below, congruent to 1 modulo 2010?
   - Why does this lead to a contradiction?

   Note below that $p_1, \ldots, p_r$ are the primes that are NOT congruent to 1 modulo 2010, so this includes 2, 3, 5, 7, and many more of your favorite primes. Also, the number 2010 in the definition of N below is not really necessary, but it should make your explanation a little easier.

   Proof. Assume that there are only finitely many primes $\not\equiv 1 \pmod{2010}$. Let $p_1, \ldots, p_r$ be the complete list of such primes. Let

   $$N = 2010 \cdot p_1 \cdots p_r - 1.$$

   We first show that all the prime divisors of N are congruent to 1 modulo 2010. To see this, it suffices to show that any prime $p \not\equiv 1 \pmod{2010}$ does not divide N. Since $p_1, \ldots, p_r$ is the complete list of such primes, this means that we want to show that $N \not\equiv 0 \pmod{p_j}$ for each $j \in \{1, 2, \ldots, r\}$. For each such j,

   $$N \equiv 0 - 1 \equiv -1 \not\equiv 0 \pmod{p_j},$$

   giving the desired result.

   Note that a product of numbers congruent to 1 modulo 2010 is itself congruent to 1 modulo 2010. Since every prime factor of N is 1 modulo 2010, we deduce then that $N \equiv 1 \pmod{2010}$. This leads to a contradiction since $N = 2010 \cdot p_1 \cdots p_r - 1$ implies $N \equiv -1 \not\equiv 1 \pmod{2010}$. Hence, our assumption is wrong, and we deduce that there are infinitely many primes $\not\equiv 1 \pmod{2010}$.