Math 580: Quiz 11

Show ALL Work Name ______Key______

1. Let α_1, α_2, and α_3 be the roots of $x^3 - 2x^2 - 3x + 4 = 0$. Calculate

$$S_k = \sum_{j=1}^{3} \alpha_j^k \quad \text{for } k = 1, 2 \text{ and } 3.$$

Make sure your work is clear enough for me to see where your answers are coming from.

$S_1 = \boxed{2}$

$S_2 = \boxed{10}$

$S_3 = \boxed{14}$

In general, if $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, with $a_n \neq 0$, has roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, then $\sigma_j = (-1)^j a_{n-j}/a_n$. What is important here is that

$$\boldsymbol{\sigma_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_n}$$

$$\boldsymbol{\sigma_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots + \alpha_{n-1}\alpha_n.}$$

Thus, $S_1 = \sigma_1 = -(-2)/1 = 2$ in this problem. As we have discussed in class, we also get from the definition of σ_1 and σ_2 that $S_2 = \sigma_1^2 - 2\sigma_2$. Since $\sigma_1 = 2$ and $\sigma_2 = (-3)/1 = -3$, we obtain $S_2 = 2^2 - 2(-3) = 10$. Finally, we use that each α_j is a root of $x^3 - 2x^2 - 3x + 4 = 0$. Thus,

$$\boldsymbol{\alpha_j^3 - 2\alpha_j^2 - 3\alpha_j + 4 = 0 \implies \alpha_j^3 = 2\alpha_j^2 + 3\alpha_j - 4.}$$

Hence,

$$\boldsymbol{\begin{aligned} S_3 &= \alpha_1^3 + \alpha_2^3 + \alpha_3^3 = (2\alpha_1^2 + 3\alpha_1 - 4) + (2\alpha_2^2 + 3\alpha_2 - 4) + (2\alpha_3^2 + 3\alpha_3 - 4) \\ &= 2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) + 3(\alpha_1 + \alpha_2 + \alpha_3) - 12 \\ &= 2S_2 + 3S_1 - 12 = 20 + 6 - 12 = 14. \end{aligned}}$$

2. The graphs of $y = x^3 - 3x^2 - 3x + 3$ and $y = x^2 + 2$ intersect in three points (x_1, y_1), (x_2, y_2) and (x_3, y_3). Calculate the value of $x_1 + x_2 + x_3$. Make sure your work is clear enough for me to see where your answer is coming from. (Comment: This should be very similar to a homework problem done in class on Monday.)

$x_1 + x_2 + x_3 = \boxed{4}$

The numbers x_1, x_2 and x_3 satisfy the equation

$$\boldsymbol{x^3 - 3x^2 - 3x + 3 = x^2 + 2.}$$

In other words, they are roots of $x^3 - 4x^2 - 3x + 1$. Thus, $x_1 + x_2 + x_3$ is the sum of the roots of this cubic. Hence, $x_1 + x_2 + x_3 = -(-4)/1 = 4$.