Math 580/780I Notes 8 Appendix

The following contains some useful information on primes of some general importance. In particular, some of the arguments (for example, that there exist arbitrarily long lists of consecutive non-squarefree numbers and that the there exist squares that are arbitrarily large containing no visible lattice points) from Notes 8 make use of the first result below.

- **Theorem 1′.** *There exist infinitely many primes.*

- **Proof 1 (Euclid's).** Assume there are only finitely many primes, say $p_1, \dots, p_r$. Then the number $p_1 \cdots p_r + 1$ is not divisible by any of the primes $p_1, \dots, p_r$, contradicting the Fundamental Theorem of Arithmetic.

- **Proof 2.** The Fermat numbers $F_n = 2^{2^n} + 1$ are odd numbers > 1 satisfying

$$F_{n+1} - 2 = \prod_{j=0}^{n} F_j.$$

Hence, they are relatively prime, so there must exist infinitely many primes.

- **Theorem 2′.** *The sum of the reciprocal of the primes diverges.*

- **Proof.** Assume $\displaystyle\sum_{p \text{ prime}} \frac{1}{p}$ converges. Then there is an N such that $\displaystyle\sum_{\substack{p \text{ prime} \\ p>N}} \frac{1}{p} < \frac{1}{2}$. Let $p_1, \dots, p_r$ denote the primes $\le N$. These primes, r and N are fixed throughout the rest of the argument. Let L be a large integer. The number of positive integers $\le 2L$ divisible by a prime $> N$ is

$$\sum_{\substack{p \text{ prime} \\ p>N}} \frac{2L}{p} < \frac{2L}{2} = L.$$

The remaining $\ge L$ positive integers $\le 2L$ can only have the prime factors $p_1, \dots, p_r$. If n is one of these L numbers, then

$$n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r} \le 2L,$$

where each e_j is a nonnegative integer. Since each $p_j^{e_j} \le 2L$, we see that $e_j \le \log(2L)/\log(p_j) \le \log(2L)/\log 2$. Since each e_j satisfies $0 \le e_j \le \log(2L)/\log 2$, the number of n of the form $p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$ above is

$$\le \left(\frac{\log(2L)}{\log 2} + 1 \right)^r \le \left(\frac{2\log(2L)}{\log 2} \right)^r \le \left(\frac{2\log(L^2)}{\log 2} \right)^r = \left(\frac{4}{\log 2} \log L \right)^r.$$

Recall that we had L such numbers, so

$$\left(\frac{4}{\log 2} \log L \right)^r \ge L \implies \frac{(\log L)^r}{L} \ge \left(\frac{\log 2}{4} \right)^r.$$

This is true for every large L. Recall r is fixed. The last inequality implies

$$\lim_{L \to \infty} \frac{(\log L)^r}{L} \ge \left(\frac{\log 2}{4} \right)^r,$$

provided the limit exists (or use limsup intead of lim here). This contradicts that the above limit is 0 (by applying L'Hôpital's rule r times). Hence, the sum of the reciprocals of the primes diverges.

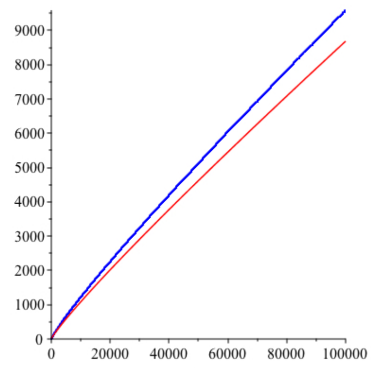
• **Comment:**. Despite Theorem $2'$, the sum of the reciprocals of every prime ever written down or printed is < 4.

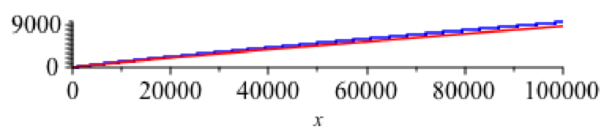
• Notation. The number of primes up to x is denoted $\pi(x)$. The next result is known as the Prime Number Theorem. In the late 1800's and early 1900's, it was commonly thought that any proof of the Prime Number Theorem must make use of complex numbers. However, it is now known that this is not the case. Nevertheless, the proof of the following is beyond the scope of this course.

• **Theorem 3′.** *The value of* $\pi(x)$ *satisfies*

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1.$$

• Below are plots of $y = \pi(x)$ in blue and $y = x/\log x$ (in red), with the second one depicting the plots with the units in the x and y direction the same.





Homework:

(1) Give a proof of Theorem $1'$ by showing that, for every integer $n > 1$, there is a prime $> n$. Use an argument similar to the first proof of Theorem $1'$ but that uses $n! + 1$ rather than $p_1 \cdots p_r + 1$.

(2) Explain why there are about the same number of primes between x and $2x$ as there are primes up to x. More precisely, show that Theorem $3'$ implies

$$\lim_{x\to\infty} \frac{\pi(2x) - \pi(x)}{\pi(x)} = 1.$$

Challenge Problem 1:

Using Theorem $2'$, show that there are at least $\sqrt{N}$ primes up to N for infinitely many positive integers N. Hint: Assume otherwise. Then, for every large positive integer k, the number of primes in the interval $(2^k, 2^{k+1}]$ is $\le \sqrt{2^{k+1}} = \sqrt{2}^{\,k+1}$. What does this mean about the sum of the reciprocals of the primes in the interval $(2^k, 2^{k+1}]$? Finally, note that

$$\sum_{p \text{ prime}} \frac{1}{p} = \sum_{k=0}^{\infty} \Bigg(\sum_{\substack{p \text{ prime} \\ 2^k < p \le 2^{k+1}}} \frac{1}{p} \Bigg).$$