Math 580/780I Notes 1

What is this course about?:

- Elementary Number Theory is the study of numbers, and in particular the study of the set of positive integers.
- Does "elementary" mean "easy"? No.
- **Example.** Consider a positive integer $m < 10^5$, and view it as a four digit number (with possible leading digit 0). Suppose all four digits are distinct. Let k be the number obtained by putting the digits of m in increasing order, and let ℓ be the number obtained by putting the digits in decreasing order. Let $m' = k - \ell$. Now repeat the process with m' in place of m. Continue. What happens? How can this be explained?

Rational and Irrational Numbers:

- Define them.
- **Theorem 1.** *$\sqrt{2}$ is irrational.*
- Give typical proof.
- **Theorem 2.** *An irrational number to an irrational power can be rational.*
- **Proof:** Consider $\sqrt{2}^{\sqrt{2}}$ and $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$.
- **Theorem 3.** *e is irrational.*
- **Proof:** Assume $e = a/b$ with a and b positive integers, and set

$$\theta = b!e - \sum_{j=0}^{b} \frac{b!}{j!} = \sum_{j=b+1}^{\infty} \frac{b!}{j!}. \tag{$*$}$$

Then

$$0 < \theta < \sum_{j=1}^{\infty} \frac{1}{(b+1)^j} = \frac{1}{b} \le 1.$$

On the other hand, the middle expression in $(*)$ is an integer. Hence, we have a contradiction and e is irrational. ■

- **Open Problem.** Is π^e irrational?
- **Open Problem.** Is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$ irrational?

Homework:

(1) Prove that $\sqrt{3}$ is irrational. Give an argument similar to that given for $\sqrt{2}$.

(2) Prove that $\sqrt{6}$ is irrational.

(3) Prove that $\sqrt{2}+\sqrt{3}$ is irrational.

(4) Prove that $\log_2 3$ is irrational.

(5) Let $I = \mathbb{R} - \mathbb{Q}$ denote the set of irrational numbers. Determine whether each of the following is true or false. If it is true, simply state so. If it is false, state so and give a counterexample.

(a) $\alpha \in I$ and $\beta \in I$ implies $\alpha + \beta \in I$

(b) $\alpha \in I$ and $\beta \in I$ implies $\alpha\beta \in I$

(c) $\alpha \in \mathbb{Q} - \{0\}$ and $\beta \in I$ implies $\alpha + \beta \in I$ and $\alpha\beta \in I$

(d) $\alpha \in I$ and $\beta \in \mathbb{Q} - \{0\}$ implies $\alpha^\beta \in I$

(e) $\alpha \in \mathbb{Q} - \{1\}$ and $\beta \in I$ implies $\alpha^\beta \in I$

Challenge Problem:

Prove that e^2 is irrational using an argument similar to that given above for e.