MATH 574: TEST 1

Name ______________________________

Instructions and Point Values: Put your name in the space provided above. Make sure that your test has seven different pages including one blank page. Work each problem below using complete English sentences in your solutions. Calculators are NOT permitted on this test.

Point Values: Problems (1) and (2) are worth 14 points each, and Problems (3), (4), (5), and (6) are worth 16 points each.

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(1) Suppose we want to use induction to prove

$$1^{1999} + 2^{1999} + 3^{1999} + \cdots + n^{1999} \geq \frac{n^{2000}}{2^{1999}} \qquad \text{for every positive integer } n.$$

(a) What would be the "induction hypothesis" in the proof? Give a complete sentence.

(b) After stating the induction hypothesis in the proof, what should the goal be? In other words, what should we be trying to establish? Be precise.

(2) Using that $\sqrt{2}$ is irrational, prove that an irrational number to an irrational power can be rational by completing the proof in the space provided. (You must give a proof by contradiction making use of what appears below.)

Proof. Assume that every irrational number to every irrational power is irrational. Since $\sqrt{2}$ is irrational, then $\sqrt{2}^{\sqrt{2}}$ is an irrational number raised to an irrational power. By our assumption, we deduce that $\sqrt{2}^{\sqrt{2}}$ is irrational. Hence, both $\sqrt{2}^{\sqrt{2}}$ and $\sqrt{2}$ are irrational numbers. By our assumption, we deduce now that

(3) Prove that $\sqrt{3}$ is irrational.

(4) The purpose of this problem is to prove that

$$2 \times 6 \times 10 \times \cdots \times (4n-2) = (n+1)(n+2)(n+3)\cdots(2n).$$

for all positive integers n. In other words, we want to show that

$$(*) \qquad \prod_{k=1}^{n}(4k-2) = \prod_{k=1}^{n}(n+k)$$

for all integers $n \geq 1$. Complete the proof given below by filling in the boxes.

Proof. We prove $(*)$ holds for all positive integers n by induction on n. Since $\prod_{k=1}^{1}(4k-2) = 2$ and $\prod_{k=1}^{1}(1+k) = 2$, we see that $(*)$ holds when $n = \boxed{\quad}$. Suppose that $(*)$ holds for some positive integer n. We show next that $(*)$ holds when n is replaced by $\boxed{\qquad\qquad}$. In other words, we want to show that

$$(**) \qquad \boxed{\qquad\qquad\qquad} = \prod_{k=1}^{n+1}(n+1+k).$$

Observe that

$$\boxed{\qquad\qquad\qquad} = \left(\prod_{k=1}^{n}(4k-2)\right) \times (4(n+1)-2) = \left(\prod_{k=1}^{n}(4k-2)\right) \times (4n+2).$$

By our induction hypothesis, we obtain

$$\prod_{k=1}^{n+1}(4k-2) = \left(\prod_{k=1}^{n}\boxed{\qquad\qquad\qquad}\right) \times (4n+2).$$

We use that

$$\left(\prod_{k=1}^{n}\boxed{\qquad\qquad\qquad}\right) = (n+1)(n+2)(n+3)\cdots(2n)$$

and $4n+2 = 2 \times (2n+1)$. Hence,

$$\prod_{k=1}^{n+1}(4k-2) = (n+1) \times (n+2) \times (n+3) \times \cdots \times (2n) \times 2 \times (2n+1).$$

Reordering the factors on the right, we obtain

$$\begin{aligned}\prod_{k=1}^{n+1}(4k-2) &= (n+2)\times(n+3)\times\cdots\times(2n)\times(2n+1)\times 2\times(n+1)\\ &= (n+2)\times(n+3)\times\cdots\times(2n)\times(2n+1)\times(2n+2)\\ &= \prod_{k=1}^{n+1}\boxed{}.\end{aligned}$$

Thus, $(**)$ holds, and we deduce by induction that $(*)$ holds for all integers $n \geq 1$. ■

(5) Using a proof by contradiction, prove that if n is an integer ≥ 2 and if $P_1, P_2, \ldots, P_n$ are n points ordered clockwise on a circle of radius 1, then the distance between two of these points must be $< 7/n$ (i.e., there exist i and j with $1 \leq i < j \leq n$ such that the distance from P_i to P_j is $< 7/n$).

Hint 1: A circle of radius 1 has circumference 2π and $2\pi < 7$.

Hint 2: The shortest path from a point A to a point B is the path along the line segment joining A to B. Draw a picture and consider the line segments $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\overline{P_3P_4}$, ..., $\overline{P_{n-1}P_n}$, and finally $\overline{P_nP_1}$.

(6) Let $a_0 = 4$, $a_1 = 5$, and

$$a_{n+1} = 3a_n - 2a_{n-1} \qquad \text{for every integer } n \geq 1.$$

For example, $a_2 = 3a_1 - 2a_0 = 15 - 8 = 7$ and $a_3 = 3a_2 - 2a_1 = 21 - 10 = 11$. Using induction, prove

$$a_n = 2^n + 3$$

for every integer $n \geq 0$.