Problem: Show that $\int_0^{\pi/2} \cos^{2n+1} x\, dx = \prod_{j=1}^{n} \left(\frac{2j}{2j+1}\right)$ for every integer $n \geq 1$.

Solution. We prove

$$\int_0^{\pi/2} \cos^{2n+1} x\, dx = \prod_{j=1}^{n} \left(\frac{2j}{2j+1}\right) \tag{$*$}$$

for every integer $n \geq 1$ by induction on n. To see that $(*)$ holds for $n = 1$, we make use of $\cos^2 x = 1 - \sin^2 x$ followed by the substitution $u = \sin x$ to see that

$$\begin{aligned}
\int_0^{\pi/2} \cos^3 x\, dx &= \int_0^{\pi/2} \cos^2 x \cos x\, dx = \int_0^{\pi/2} (1 - \sin^2 x) \cos x\, dx \\
&= \int_0^1 (1 - u^2)\, du = u - \frac{u^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3} = \prod_{j=1}^{1} \left(\frac{2j}{2j+1}\right).
\end{aligned}$$

Next, we suppose that $(*)$ holds for some positive integer $n = k \geq 1$. Thus,

$$\int_0^{\pi/2} \cos^{2k+1} x\, dx = \prod_{j=1}^{k} \left(\frac{2j}{2j+1}\right). \tag{$**$}$$

Next, we show that $(*)$ holds for $n = k + 1$. To clarify, we want to show that

$$\begin{aligned}
\int_0^{\pi/2} \cos^{2k+3} x\, dx = \int_0^{\pi/2} \cos^{2(k+1)+1} x\, dx &= \prod_{j=1}^{k+1} \left(\frac{2j}{2j+1}\right) \\
&= \prod_{j=1}^{k} \left(\frac{2j}{2j+1}\right) \times \frac{2(k+1)}{2(k+1)+1} = \frac{2k+2}{2k+3} \prod_{j=1}^{k} \left(\frac{2j}{2j+1}\right).
\end{aligned}$$

To show this, we begin by applying integration by parts with $u = \cos^{2k+2} x$ and $v = \sin x$ to deduce

$$\begin{aligned}
\int_0^{\pi/2} \cos^{2k+3} x\, dx &= \int_0^{\pi/2} (\cos^{2k+2} x)(\cos x)\, dx = \int_{x=0}^{x=\pi/2} u\, dv = u\, v\Big|_{x=0}^{x=\pi/2} - \int_{x=0}^{x=\pi/2} v\, du \\
&= (\cos^{2k+2} x)(\sin x)\Big|_0^{\pi/2} - \int_0^{\pi/2} (\sin x)(2k+2)(\cos^{2k+1} x)(-\sin x)\, dx \\
&= \int_0^{\pi/2} (\sin^2 x)(2k+2)(\cos^{2k+1} x)\, dx = (2k+2) \int_0^{\pi/2} (1 - \cos^2 x)(\cos^{2k+1} x)\, dx \\
&= (2k+2) \int_0^{\pi/2} (\cos^{2k+1} x)\, dx - (2k+2) \int_0^{\pi/2} (\cos^{2k+3} x)\, dx.
\end{aligned}$$

Adding $(2k+2)\int_0^{\pi/2}(\cos^{2k+3}x)\,dx$ to the first and last parts of these equations gives

$$(2k+3)\int_0^{\pi/2}(\cos^{2k+3}x)\,dx = (2k+2)\int_0^{\pi/2}(\cos^{2k+1}x)\,dx.$$

Combining this with $(**)$ gives

$$\int_0^{\pi/2}(\cos^{2k+3}x)\,dx = \frac{2k+2}{2k+3}\int_0^{\pi/2}(\cos^{2k+1}x)\,dx = \frac{2k+2}{2k+3}\prod_{j=1}^{k}\left(\frac{2j}{2j+1}\right).$$

This is what we said we wanted to show, so we deduce now that $(*)$ holds for $n = k+1$. Thus, by induction, $(*)$ holds for every integer $n \geq 1$. ■