# Some Answers & Some Solutions to Old Math 574 Final Exams

**Part I, Fall, 1999**

1. $a_n = 2^n + 1$ for $n \geq 0$

2. 97

3. We prove $a_n \leq 2$ for all $n \geq 1$ by induction. Observe that $a_1 = \sqrt{2} \leq 2$. Suppose $k$ is a positive integer for which $a_k \leq 2$ (this is our induction hypothesis). Then $a_{k+1} = \sqrt{2}^{a_k} \leq \sqrt{2}^2 = 2$. Hence, $a_n \leq 2$ for all $n \geq 1$ by induction.

4. $OKLMFBDECAIHGPJN$ (one of many correct answers)

5. Fix one of the six points, and call it $A$. Five of the 15 line segments have $A$ as an endpoint. Each is colored red or blue. By the pigeonhole principle, three (or more) of these five line segments must be colored the same color, say blue (if three are colored red, do a similar argument to what follows but with the words "blue" and "red" interchanged). Suppose the other endpoints of three edges colored blue are $B$, $C$, and $D$ (so each of $\overline{AB}$, $\overline{AC}$, and $\overline{AD}$ is blue). If any one of the edges of $\triangle BCD$ is colored blue, then there would be a monochromatic blue triangle using that edge and two of $\overline{AB}$, $\overline{AC}$, and $\overline{AD}$. On the other hand, if each edge of $\triangle BCD$ is colored red, then there is a monochromatic red triangle, namely $\triangle BCD$. Thus, in any case, there must be a monochromatic triangle.

6. $n2^{n-1}$

7. 231

8. You should explain why $A_n$ satisfies the recursion $A_n = 3A_{n-1} + 4A_{n-2}$. The answer is $\left(4^{n+1} + (-1)^n\right)/5$.

**Part II, Fall, 1999**

1. (a) Second
   (b) Remove 2 coins from the stack of size 11.

2. Assume $\log_{10} 24$ is rational. Then $\log_{10} 24 = a/b$ where $a$ and $b$ are positive integers. We deduce that $24 = 10^{a/b}$. Hence, $24^b = \left(10^{a/b}\right)^b = 10^a$. Since $10^a$ ends in a 0 and $24^b$ does not (it's not divisible by 5), we get a contradiction. Therefore, $\log_{10} 24$ is irrational.

3. $-32$ (plug in $x = 1$)

4. (a) 6
   (b) 720

5. The letter "$A$" should be placed in the square with the circled "$G$".

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| B | B | B | B | B | B | B | B | B | S |
| B | B | B | G | B | B | B | B | B | B |
| B | B | B | B | B | B | B | G | B | B |
| B | B | B | B | B | G | B | B | B | B |
| B | G | B | B | B | B | B | B | B | B |
| B | B | G | B | B | B | B | B | B | B |
| G | B | B | B | B | B | B | B | B | B |

**Part I, Spring, 1999**

1. Assume $\log_2 3$ is rational. Then there are positive integers $a$ and $b$ such that $\log_2 3 = a/b$. We deduce that $3 = 2^{a/b}$. Hence, $3^b = \left(2^{a/b}\right)^b = 2^a$. Since $3^b$ is odd and $2^a$ is even, we have a contradiction. Therefore, $\log_2 3$ is irrational.

2. We use induction to show that the sum of the first $n$ odd numbers is $n^2$. More precisely, we want to show

$$1 + 3 + 5 + \cdots + (2n-1) = n^2 \qquad (*)$$

for all positive integers $n$. When $n = 1$, $(*)$ is true as $1 = 1^2$. Suppose $(*)$ holds for $n = k$ where $k$ is some positive integer. Then

$$1 + 3 + 5 + \cdots + (2k-1) + (2k+1) = k^2 + 2k + 1 = (k+1)^2.$$

Note that $2n - 1 = 2k + 1$ when $n = k + 1$. We deduce that $(*)$ holds for $n = k + 1$. Hence, $(*)$ holds for every positive integer $n$ by induction.

3. We prove that

$$f_{3n} \text{ is even and } f_{3n-1} \text{ and } f_{3n-2} \text{ are odd} \qquad (*)$$

for every integer $n \geq 1$ by induction on $n$. Since $f_3 = 2$, $f_2 = 1$, and $f_1 = 1$, $(*)$ holds for $n = 1$. Suppose $(*)$ holds for $n = k$. Then $f_{3k+1} = f_{3k} + f_{3k-1}$ is an even number plus and an odd number, so $f_{3k+1}$ is odd. Now, $f_{3k+2} = f_{3k+1} + f_{3k}$ is an odd number plus and an even number, so $f_{3k+2}$ is odd. Also, $f_{3k+3} = f_{3k+2} + f_{3k+1}$ is an odd number plus and an odd number, so $f_{3k+3}$ is even. Note that for $n = k + 1$, we have $3n = 3k + 3$, $3n - 1 = 3k + 2$, and $3n - 2 = 3k + 1$. Thus, $(*)$ holds for $n = k + 1$. Therefore, $(*)$ holds for every $n \geq 1$ by induction on $n$.

4. 3

5. $2^n$

6. 146

7 & 8. See Problems 6 & 7 from Part I, Fall, 1999.

**Part II, Spring, 1999**

1. (a) First
   (b) 4

2. Let "O" stand for "an odd number" and "E" stand for "an even number". A lattice point $(x, y, z)$ in space has 8 possible forms: $(\mathrm{E}, \mathrm{E}, \mathrm{E})$, $(\mathrm{E}, \mathrm{E}, \mathrm{O})$, $(\mathrm{E}, \mathrm{O}, \mathrm{E})$, $(\mathrm{E}, \mathrm{O}, \mathrm{O})$, $(\mathrm{O}, \mathrm{E}, \mathrm{E})$, $(\mathrm{O}, \mathrm{E}, \mathrm{O})$, $(\mathrm{O}, \mathrm{O}, \mathrm{E})$, and $(\mathrm{O}, \mathrm{O}, \mathrm{O})$. By the pigeonhole principle, if there are 9 points in space, two of them must have the same form. Let $(a, b, c)$ and $(d, e, f)$ be 2 such lattice points. Since the sum of two even numbers is even and the sum of two odd numbers is even, each of $a + d$, $b + e$, and $c + f$ is even. Hence, $(a + d)/2$, $(b + e)/2$, and $(c + f)/2$ are integers. Since these are the coordinates of the midpoint of $(a, b, c)$ and $(d, e, f)$, we deduce that this midpoint is a lattice point.

3. The first graph is not complete, is not connected, and is planar (so the first three answers are "No", "No", and "Yes"). The second graph is not complete, is connected, and is not planar (so the next three answers are "No", "Yes", and "No"). The diameter of the second graph is 2.

4. We show that there is a connection between this problem and a problem about graphs that we have already done. Represent the six people by vertices $A, B, C, D, E$, and $F$. Consider the complete graph on these six vertices. Also, consider all the handshakes that took place. If two people have shook hands, color the edge between the two people (the two corresponding vertices) red. If two people have not shook hands, then color the edge between them blue. Since each edge of the complete graph on 6 vertices is colored either red or blue, we deduce that there must be a monochromatic triangle (see Problem 5 from Part I, Fall, 1999). Hence, there must either be three people any two of which shook hands with each other or three people any two of which did not shake hands with each other.

5. The letter "$A$" should be placed in the square with the circled "$G$".

| | | | | | | |
|---|---|---|---|---|---|---|
| B | B | B | G | B | B | |
| B | B | G | B | B | B | B |
| B | G | B | B | B | B | B |
| G | B | B | B | B | B | B |

6. Every time a handshake occurs, two people shake a hand. Therefore, if we add up the number of hands each person shakes, then we get twice the number of handshakes that have taken place. Hence, if we take the number of hands each person shakes and total all these numbers, we end up with an even number. Since the sum of an odd number of odd numbers is odd, we must have an even number of odd numbers if the total of the numbers is even. We deduce that there must be an even number of people who have shaken an odd number of hands at the party.

7. See Problem 8 from Part I, Fall, 1999.