MATH 574: TEST 1

Name ______________________________

Instructions and Point Values: Put your name in the space provided above. Make sure that your test has seven different pages including one blank page. Work each problem below using complete English sentences in your solutions. Calculators are <u>NOT</u> permitted on this test.

Point Values: Problems (1) and (2) are worth 14 points each, and Problems (3), (4), (5), and (6) are worth 18 points each.

---

(1) Suppose we want to use induction to prove

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \geq \ln(n+1)$$

for all positive integers n.

(a) What would be the "induction hypothesis" in the proof?

(b) After stating the induction hypothesis in the proof, what should the goal be? In other words, what should we be trying to establish? Be precise.

(2) Using that $\sqrt{2}$ is irrational, prove that an irrational number to an irrational power can be rational by completing the proof in the space provided. (You must give a proof by contradiction making use of what appears below.)

Proof. Assume that every irrational number to every irrational power is irrational. Since $\sqrt{2}$ is irrational, then $\sqrt{2}^{\sqrt{2}}$ is an irrational number raised to an irrational power. By our assumption, we deduce that $\sqrt{2}^{\sqrt{2}}$ is irrational. Hence, both $\sqrt{2}^{\sqrt{2}}$ and $\sqrt{2}$ are irrational numbers. By our assumption, we deduce now that

(3) Prove that $\log_2 3$ is irrational.

(4) The purpose of this problem is to prove that

$$(*) \qquad 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all positive integers n. Complete the proof given below by filling in the boxes.

Proof. We prove $(*)$ holds for all positive integers n by induction on n. When n is replaced by 1, the fraction on the right-hand side of $(*)$ equals 1. Since $1^2 = 1$, we see that $(*)$ holds when $n = \boxed{}$. Suppose that $(*)$ holds for some positive integer n. We show next that $(*)$ holds when n is replaced by $\boxed{}$. In other words, we show that

$$(**) \qquad 1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \boxed{}$$

By our induction hypothesis, we obtain

$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \boxed{} + (n+1)^2.$$

We factor out $n+1$ from both expressions on the right-hand side to obtain

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 &= (n+1) \times \left(\boxed{} + (n+1) \right) \\ &= (n+1) \times \left(\frac{2n^2 + 7n + 6)}{6} \right) \\ &= \boxed{}. \end{aligned}$$

Thus, $(**)$ holds, and we deduce by induction that $(*)$ holds for all positive integers n. ■

(5) Prove that

$$x + y \leq \sqrt{2(x^2 + y^2)}$$

for all real numbers x and y.

(6) Let $a_1 = \sqrt{6}$, and let

$$a_{n+1} = \sqrt{6 + a_n} \quad \text{for each positive integer } n.$$

For example, $a_2 = \sqrt{6 + a_1} = \sqrt{6 + \sqrt{6}}$ and $a_3 = \sqrt{6 + a_2} = \sqrt{6 + \sqrt{6 + \sqrt{6}}}$. Prove that

$$a_n \leq 3 \quad \text{for every positive integer } n.$$