MATH 574, NOTES 9
GRAPHS AND TREES

▶ **Definition:** A *graph* G is a finite collection of vertices V together with a collection of edges E (each edge joins two elements of V). We write $G = (V, E)$. We consider only the case where each pair of vertices in V corresponds to at most one edge in E (but possibly none). We denote each element of E as an unordered pair $\{u, v\}$ where both u and v are in V and suppose that $u \neq v$ for each $\{u, v\} \in E$.

▶ **Definitions:** A *path* from $u \in V$ to $v \in V$ in a graph $G = (V, E)$ is a sequence of edges $e_j = \{x_{j-1}, x_j\} \in E$, where $j \in \{1, 2, \ldots, r\}$, such that $x_0 = u$ and $x_r = v$. A *cycle* in a graph G is a path from u to u consisting of distinct edges $e_j = \{x_{j-1}, x_j\} \in E$ for $j \in \{1, 2, \ldots, r\}$ with $x_0 = x_r = u$ and with $x_0, x_1, \ldots, x_{r-1}$ distinct. A *connected* graph is a graph having the property that there is a path from u to v for every choice of u and v from its vertex set V. The *distance* between two vertices u and v in a connected graph is the minimal number of edges in any path from u to v.

▶ **Definition:** A *tree* is a connected graph which contains no cycle.

Examples:

(1) Illustrate the above definitions with examples.

(2) What do family trees have to do with trees?

(3) A *complete* graph on n vertices is a graph $G = (V, E)$ with $|V| = n$ and such that for every choice of u and v in V, there is an edge $\{u, v\} \in E$. How many edges are there in a complete graph on n vertices?

(4) If each edge of a complete graph on 6 vertices is colored either red or blue, why must the graph contain a monochromatic triangle (i.e., why must there exist 3 vertices A, B, and C such that each of the edges $\{A, B\}$, $\{A, C\}$, and $\{B, C\}$ is colored the same)?

(5) If each edge of a complete graph on 17 vertices is colored either red, blue, or green, why must the graph contain a monochromatic triangle?

(6) Let u and v be vertices in a graph $G = (V, E)$. Prove that if there are two or more distinct paths from u to v each containing no repeated edges (or vertices), then there is a cycle in G. What does this imply about trees?

(7) How many edges does a tree on n vertices have?

(8) For u a vertex in a graph $G = (V, E)$, let the *degree* of u (written $\deg(u)$) denote the number of edges containing u. Prove that a tree with $n \geq 2$ vertices has at least two vertices with degree 1. (Hint: Consider u and v with the distance between them maximal, and show each of u and v has degree 1.)

(9) An *Euler path* in a graph G is a path that traverses every edge of G exactly once. Give examples

of graphs where Euler paths exist and other examples where they do not exist. Discuss a necessary and sufficient condition for the existence of an Euler path.

(10) A graph is called *planar* if it can be drawn in a plane without any edges intersecting or vertices overlapping. Give examples of graphs which are planar and other graphs which are not. Discuss a necessary and sufficient condition for a graph to be planar.