MATH 574, NOTES 8
RECURRENCE RELATIONS

▶ **Definition:** A *recurrence relation* for a sequence $\{a_n\}$ is a formula which expresses the nth term of the sequence in terms of one or more of the previous terms of the sequence (when $n \geq n_0$ for some n_0).

▶ **Comment:** Typically the recurrence relation is known and one wants to determine explicitly the sequences which satisfy the given recurrence relation.

Examples:

(1) Given $a_1 = 1$ and $a_n = 2a_{n-1}$ for every $n \geq 2$, determine a_n.

(2) A person deposits \$1000 into a savings account earning 2% interest compounded annually and never withdraws or adds money to the account (other than the interest earned). How much will be in the account after n years?

(3) The Tower of Hanoi Problem. Describe it and solve it.

(4) Linear recurrences (with constant coefficients). Define them and describe how to solve one using its *characteristic equation.*

(5) Find a general formula for the nth Fibonacci number.

(6) Suppose $u_0 = 0$, $u_1 = 1$, and $u_n = u_{n-1} + 2u_{n-2}$ for $n \geq 2$. What is the characteristic equation for this recursion? What are its roots? Show that $u_n = \frac{1}{3}\big(2^n - (-1)^n\big)$ for all $n \geq 0$.

(7) Suppose $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, and $a_n = 3a_{n-2} + 2a_{n-3}$ for $n \geq 3$. Show that $a_n = \frac{1}{9}\big((3n-4)(-1)^n + 2^{n+2}\big)$ for all $n \geq 0$.

(8) Suppose $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, and $a_n = -2a_{n-1} - a_{n-2} - 2a_{n-3}$ for $n \geq 3$. Show that

$$a_n = \left(\frac{-2-9\mathbf{i}}{10}\right)\mathbf{i}^n + \left(\frac{-2+9\mathbf{i}}{10}\right)(-\mathbf{i})^n + \frac{2}{5}(-2)^n \quad \text{for all } n \geq 0.$$

(9) Suppose $u_0 = 4$, $u_1 = 5$, and $u_n = 3u_{n-1} - 2u_{n-2}$ for $n \geq 2$. Find an explicit formula for u_n.

(10) Suppose $a_0 = 4$, $a_1 = 5$, and $a_n = 2a_{n-1} + 2a_{n-2}$ for $n \geq 2$. Find an explicit formula for a_n.

(11) Suppose $u_0 = 0$, $u_1 = 1$, $u_2 = 2$, and $u_n = 3u_{n-1} - 2u_{n-2} + 6u_{n-3}$ for $n \geq 3$. Find an explicit formula for u_n.