MATH 574, NOTES 3
PRACTICE PROBLEMS FOR TEST 1

(1) Prove that if the product of two positive numbers is < 100, then at least one of the numbers is < 10.

(2) We showed that $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$ for all $n \geq 1$. Using this information, prove that

$$(1 + 2 + 3 + \cdots + n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3.$$

(3) Prove that $\displaystyle\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n}$ for every integer $n \geq 1$.

(4) It is not true that the product of a rational number and an irrational number is always irrational. Prove that if α is rational and β is irrational, then $\alpha\beta$ is irrational unless α equals $\boxed{}$. Fill in the box with the correct number (there's just one) and write the proof.

(5) (a) Let $\alpha = e^{1/e}$. Suppose $a_1 = \alpha$, $a_2 = \alpha^\alpha = \alpha^{a_1}$, $a_3 = \alpha^{a_2}$, and so on. Prove that $a_n \leq e$ for all integers $n \geq 1$.

(b) Does the fact that $e = 2.71828\ldots$ have anything to do with your proof? In other words, is it true that if the number e is replaced everywhere in part (a) by any number $t > 0$, then the argument still works?

(6) (a) Complete the proof of the lemma below. (The proof is not by contradiction or induction.)

Lemma. *If a is an integer, then the remainder when a^2 is divided by 4 is either 0 or 1.*

Proof. We want to show that there is an integer q such that $a^2 = 4q + r$ with $r = 0$ or $r = 1$. The remainder when a is divided by 4 is one of 0, 1, 2, or 3. If the remainder is 0, then $a = 4k$ for some integer k so that $a^2 = 16k^2 = 4(4k^2) + 0$. Thus, in this case, one can take $q = 4k^2$ and $r = 0$. If the remainder is 1, then $a = 4k + 1$ for some integer k so that $a^2 = 16k^2 + 8k + 1 = 4(4k^2 + 2k) + 1$. In this case, one can take $q = 4k^2 + 2k$ and $r = 1$. If the remainder is 2, then

$$a = \boxed{}$$

for some integer k so that

$$a^2 = \boxed{}.$$

In this case, one can take

$$q = \boxed{} \quad \text{and} \quad r = \boxed{}.$$

If the remainder is 3, then

$$a = \boxed{}$$

for some integer k so that

$$a^2 = \boxed{}.$$

In this case, one can take

$$q = \boxed{} \quad \text{and} \quad r = \boxed{}.$$

Thus, no matter what the remainder is when a is divided by 4, we deduce that the remainder when a^2 is divided by 4 is either 0 or 1. This completes the proof. ■

(b) Prove that $N = 3420392835475334299902849348202261018908732920143$ is not the sum of two squares. In other words, show that there are no integers a and b such that $N = a^2 + b^2$. (Hint: Decide whether you want to do a proof by contradiction or a proof by induction. One of these works. Then determine the remainder when N is divided by 4. Next, use the lemma to obtain information about both a^2 and b^2 and proceed.)