MATH 532, 736I: MODERN GEOMETRY

Test 2, Spring 2013 Name ______________________________

Show All Work

Instructions: This test consists of 5 pages (one is an information page). Put your name at the top of this page and at the top of the first page of the packet of blank paper given to you. Work each problem in the packet of paper unless it is indicated that you can or are to do the work below. Fill in the boxes below with your answers. Show <u>ALL</u> of your work. Do <u>NOT</u> use a calculator.

10 pts (1) Let A and B be two different points. Let $C = (9A + B)/10$. On the blank pages provided with this test, explain why C is on line $\overleftrightarrow{AB}$. You should refer to a theorem on the last page and be explicit about what you are taking for each variable in the theorem.

12 pts (2) Let A, B, C and D be four different points (in a plane). Suppose $(A - B)^2 = (C - D)^2$, $(A - C)^2 = (B - D)^2$ and $(A - B)(A - C) = (B - A)(B - D) = 0$. Which one of the following must be true about the four points A, B, C and D? Justify your answer using the blank paper provided. Basically, I am looking for an explanation of what these equations mean geometrically. (You do not need to explain why the other choices are incorrect.)

(i) B, C and D are on a circle centered at A.

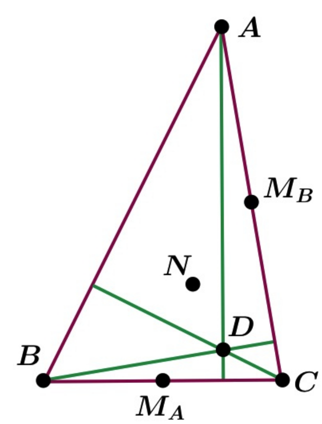
(ii) A, B, C and D are the four vertices of a square.

(iii) A, B, C and D are the four vertices of a rectangle but not necessarily of a square.

16 pts (3) Let A, B, and C be 3 noncollinear points. Let M_A be the midpoint of $\overline{BC}$, and let M_B be the midpoint of $\overline{AC}$. Let D be the intersection of the (extended) altitudes of ΔABC. Let

$$N = \frac{A + B + C + D}{4}.$$

Prove that the distance from N to M_A is the same as the distance from N to M_B. This is part of the 9-point circle theorem, so you should not make use of the 9-point circle theorem in doing this problem. Instead you want to prove this part of the theorem.



20 pts (4) Let A, B, and C be 3 noncollinear points, and let A', B', and C' be 3 noncollinear points with $A \neq A'$, $B \neq B'$, and $C \neq C'$. Suppose that the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are parallel. Suppose further that $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ intersect at some point P, that $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ intersect at some point R, and that $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ are parallel. (See the picture on the last page.) What follows is a proof that $\overleftrightarrow{BC}$ and $\overleftrightarrow{PR}$ are parallel except there are some boxes which need to be filled in. Complete the proof by filling in the boxes. There may be more than one correct way to fill in a box. The goal is to end up with a correct proof that $\overleftrightarrow{BC}$ and $\overleftrightarrow{PR}$ are parallel.

Proof: Since $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are parallel, there are real numbers k_1 and k_2 such that

$$A' - A = k_1(B' - B) = k_2(C' - C).$$

We get that

$$A - k_1 B = A' - k_1 B'.$$

We first explain why $k_1 \neq \boxed{}$.

Give explanation here: $\boxed{}$

Hence,

$$\left(\frac{1}{1-k_1}\right) A + \left(\frac{-k_1}{1-k_1}\right) B = \left(\frac{1}{1-k_1}\right) A' + \left(\frac{-k_1}{1-k_1}\right) B'.$$

By Theorem 1 (from the last page of this exam) with $t = \boxed{}$, we see that the expression on the left above is a point on line $\overleftrightarrow{AB}$ and that the expression on the right above is a point on line $\overleftrightarrow{A'B'}$. Therefore, we get that

(1) $$(1 - k_1)P = A - k_1 B.$$

Similarly, from $k_2 C - A = k_2 C' - A'$, we deduce that

(2) $\boxed{}$.

(Your answer should contain A, C, and R.)

Using that

$$k_1 B - k_2 C = k_1 B' - k_2 C'$$

and that $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ are parallel (so Q is a point at "infinity"), we obtain

(3) $\boxed{}$.

(Your answer should NOT involve Q.)

From (1) and (2), we obtain

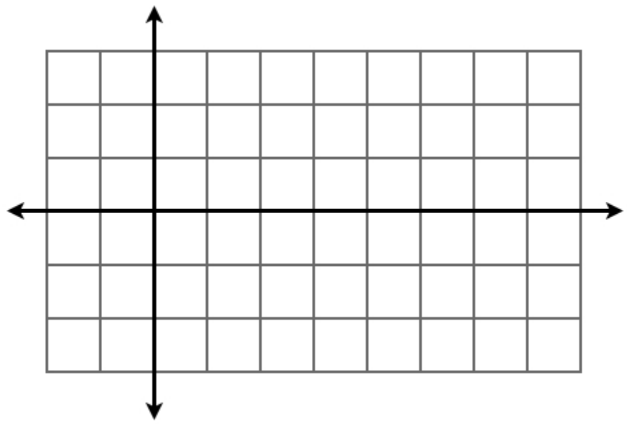
(4) $\boxed{} = k_2 C - k_1 B.$

Using (3), we can rewrite (4) in the form

$$(1 - k_1) \times \boxed{\qquad\qquad} = k_1 \times \boxed{\qquad\qquad}.$$

We deduce that the vector $\boxed{\quad}$ and the vector $\overrightarrow{BC}$ are parallel. Hence, lines $\boxed{\quad}$ and $\overleftrightarrow{BC}$ are parallel, completing the proof. ■

$\boxed{\text{20 pts}}$ (5) The figure on the right is to help you with plotting points to make computations for this problem. You can use matrices to do some of the work here, but most (maybe all) of you will likely prefer simply to plot points. For this problem,



$$f = R_{\pi,(4,0)} T_{(5,-1)} R_{\pi/2,(1,0)} R_{\pi,(1,1)}.$$

(a) Calculate $f(2,2)$. Put your answer below. For work, you may simply plot points indicating where the point $(2,2)$ goes under each rotation and translation that defines f.

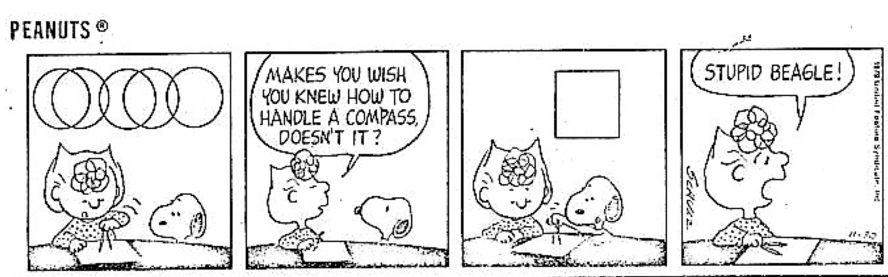
$f(2,2) = \boxed{\qquad}$

(b) In the first box below, write f as a translation T_P or a rotation $R_{\theta,(x,y)}$ (whichever it is). Here, P should be an explicit point or θ, x and y should be explicit numbers with θ in radians. Also, tell me which theorem you are using here by filling in the second box below.

$f = \boxed{\qquad}$ from using Theorem $\boxed{\quad}$ (on the last page of the test).

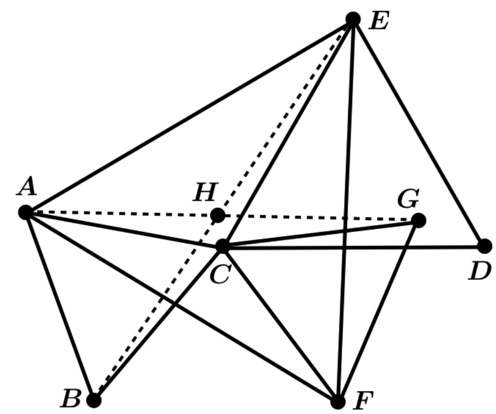
(c) Calculate the value of $f(2,-2)$ by using your answer to (b).

$f(2,-2) = \boxed{\qquad}$



22 pts

(6) In the picture to the right, triangles $\triangle ABC$, $\triangle CFG$, $\triangle ECD$ and $\triangle AFE$ are equilateral. The point H is the midpoint of segment $\overline{BE}$. Complete the parts below to show that H is also the midpoint of $\overline{AG}$.



(a) Let $f = R_{\pi/3,C}R_{\pi/3,E}R_{\pi/3,A}$. What is the value of $f(B)$? (It is a point in the picture.)

$f(B) =$

(b) With f as in part (a), we know from an important theorem in class that $f = R_{\pi,P}$ for some point P. What is the point P? (It is a point in the picture.)

$P =$

Explanation for your answer:

(c) Justify that point H is the midpoint of segment $\overline{AG}$. Be clear. Read over what you write and make sure that a reader can understand how you are showing H is the midpoint of segment $\overline{AG}$ without using or appearing to use what you are trying to show. (Hint: You calculated $f(B)$. For this part, you should be calculating a different value of f, but be sure to show the correct work for calculating this value of f. Someone reading your answer should be able to tell how you are using equilateral triangles in the picture above.)

INFORMATION PAGE

Theorem 1: *Let A and B be distinct points. Then C is a point on line $\overleftrightarrow{AB}$ if and only if there is a real number t such that $C = (1-t)A + tB$.*

Theorem 2: *If A, B and C are points and there are real numbers x, y, and z not all 0 such that*

$$x + y + z = 0 \quad \textit{and} \quad xA + yB + zC = \overrightarrow{0},$$

then A, B and C are collinear.

Theorem 3: *If A, B and C are collinear, then there are real numbers x, y and z not all 0 such that*

$$x + y + z = 0 \quad \textit{and} \quad xA + yB + zC = \overrightarrow{0}.$$

Theorem 4: *Let $\alpha_1, \ldots, \alpha_n$ be real numbers (not necessarily distinct), and let $A_1, \ldots, A_n$ and $B_1, \ldots, B_k$ be points (not necessarily distinct). Let f be a product of the n rotations R_{α_j, A_j} and the k translations T_{B_j} with each of the n rotations and k translations occurring exactly once in the product. If $\alpha_1 + \cdots + \alpha_n$ is not an integer multiple of 2π, then there is point C such that*

$$f = R_{\alpha_1+\alpha_2+\cdots+\alpha_n, C}.$$

If $\alpha_1 + \cdots + \alpha_n$ is an integer multiple of 2π, then f is a translation.

$$T_{(a,b)} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \qquad T_{(a,b)} = R_{\pi,(a/2,b/2)} R_{\pi,(0,0)}$$

$$R_{\theta,(x_1,y_1)} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & x_1(1-\cos(\theta)) + y_1 \sin(\theta) \\ \sin(\theta) & \cos(\theta) & -x_1 \sin(\theta) + y_1(1-\cos(\theta)) \\ 0 & 0 & 1 \end{pmatrix}$$

Picture for Problem 4: