MATH 532, 736I: MODERN GEOMETRY

Name ________________________________

Test #2 (1995)

Show All Work
Points: Problem (7) is worth 16 points; each of the other problems is worth 14 points.

(1) Theorems are listed on the last page of this test. They may or may not have the numbering that you are accustomed to them having from class. Prove Theorem 2 using Theorem 1 (but not Theorem 3 or Theorem 4).

(2) Let A, B, and C be 3 noncollinear points. Let D be the intersection of the (extended) altitudes of ΔABC. Let M_A be the midpoint of $\overline{BC}$, and let M_C be the midpoint of $\overline{AB}$. Let $N = (A + B + C + D)/4$. Prove that the distance from N to M_A is the same as the distance for N to M_C. This is part of the 9$-$point circle theorem, so you should not make use of the 9$-$point circle theorem in doing this problem.

(3) Let $A = (2, 1)$ and $B = (2, 2)$. Calculate each of the following, and put the answer in the appropriate box. Each answer should be either a specific number (write it down) or a specific point (write down its coordinates). Simplify your answers.

(a) $B - 2A = \boxed{}$

(b) $(B - A)^2 = \boxed{}$

(c) $A(BA) = \boxed{}$

(4) The Pythagorean Theorem states that if ΔABC is a right triangle with legs (the shorter two sides) of length a and b and with hypotenuse (the longer side) of length c, then $a^2 + b^2 = c^2$. Prove the converse of the Pythagorean Theorem by making use of vectors. (Do NOT prove the Pythagorean Theorem!!) In other words, suppose ΔABC is a given triangle with side $\overline{AB}$ of length a, side $\overline{BC}$ of length b, and side $\overline{AC}$ of length c, and suppose that $a^2 + b^2 = c^2$. Using vectors, prove that $\angle ABC$ must be a right angle. (Comment and Hint: A proof that does not make appropriate use of vectors will be worth 0 points. A correct solution should involve interpreting the equation $a^2 + b^2 = c^2$ in terms of vectors and simplifying the result to deduce that $\angle ABC$ must be a right angle.)

(5) The first picture on the second to the last page consists of 2 triangles ΔABC and $\Delta A'B'C'$ which are perspective from the line passing through R, S, and T. Suppose we want to prove that these triangles are also perspective from a point. Let X be the point of intersection of the lines $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$. To show that $\overleftrightarrow{CC'}$ also passes through X, we can consider two triangles which we know to be perspective from a point. The vertices of these two triangles (not necessarily the edges) and the point they are perspective from are all in the drawing. Determine the two triangles and the point they are perspective from and indicate your answers in the boxes below. (No work necessary.)

Triangles $\boxed{\qquad\qquad\qquad}$ and $\boxed{\qquad\qquad\qquad}$ are perspective from point $\boxed{\quad\;}$.

(Note: You can complete the boxes to make a sentence which is true, but that will not necessarily insure that you have a correct answer. A correct answer must correspond to establishing that $\overleftrightarrow{CC'}$ passes through X as discussed above.)

(6) Let A, B, and C be distinct points with the distance from A to B equal to the distance from A to C. Using vectors, explain why the line passing through A and the midpoint of line segment $\overline{BC}$ is perpendicular to line $\overleftrightarrow{BC}$. (To receive credit for this problem, you must make appropriate use of vectors. In particular, you should not use an argument which involves congruent triangles.)

(7) Let A, B, and C be 3 noncollinear points, and let A', B', and C' be 3 noncollinear points with $A \neq A'$, $B \neq B'$, and $C \neq C'$. Suppose that the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are parallel. Suppose further that $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ intersect at some point P, that $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ intersect at some point Q, and that $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ are parallel. (See the second picture on the second to the last page.) The next two pages contain a proof that $\overleftrightarrow{AC}$ and $\overleftrightarrow{PQ}$ are parallel except there are some boxes which need to be filled. Complete the proof by filling in the boxes. There may be more than one correct way to fill in a box. The goal is to end up with a correct proof that $\overleftrightarrow{AC}$ and $\overleftrightarrow{PQ}$ are parallel.

Proof: Since $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are parallel, there are real numbers k_1 and k_2 such that

$$A' - A = k_1(B' - B) = k_2(C' - C).$$

We get that

$$A - k_1 B = A' - k_1 B'.$$

We first explain why $k_1 \neq \boxed{}$.

Give explanation here: $\boxed{}$

Hence,

$$\left(\frac{1}{1-k_1}\right) A + \left(\frac{-k_1}{1-k_1}\right) B = \left(\frac{1}{1-k_1}\right) A' + \left(\frac{-k_1}{1-k_1}\right) B'.$$

By Theorem 1 (from the last page of this exam) with $t = \boxed{}$, we see that the expression on the left above is a point on line $\overleftrightarrow{AB}$ and that the expression on the right above is a point on line $\overleftrightarrow{A'B'}$. Therefore, we get that

(1) $$(1 - k_1)P = A - k_1 B.$$

Similarly, from $k_1 B - k_2 C = k_1 B' - k_2 C'$, we deduce that

(2) $$\boxed{}.$$

(Your answer should contain Q, B, and C.)

Using that

$$A - k_2 C = A' - k_2 C'$$

and that $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ are parallel (so R is a point at "infinity"), we obtain

(3) $$\boxed{}.$$

(Your answer should NOT involve R.)

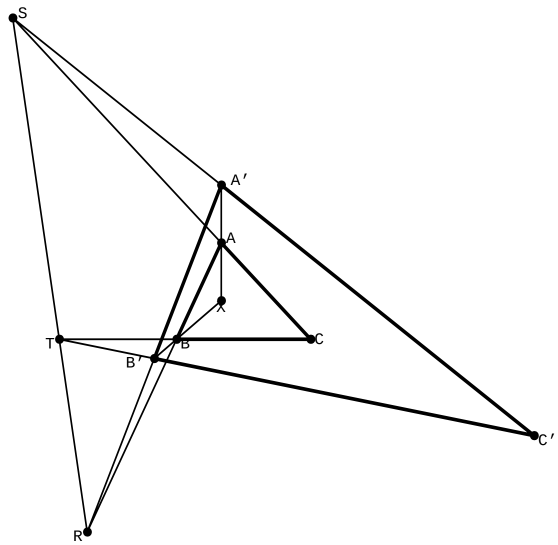
From (1) and (2), we obtain

(4) $$\boxed{} = A - k_2 C.$$

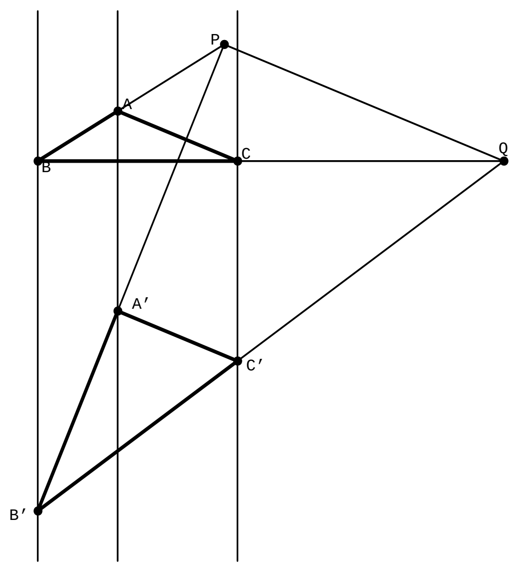
Using (3), we can rewrite (4) in the form

$$(1 - k_1) \times \boxed{} = \boxed{}.$$

Recall that $k_1 \neq 1$. Therefore, the line $\overleftrightarrow{AC}$ is parallel to the line $\overleftrightarrow{PQ}$.



Problem 5



Problem 7

INFORMATION PAGE

Theorem 1: *Let A and B be distinct points. Then C is a point on line $\overleftrightarrow{AB}$ if and only if there is a real number t such that*

$$C = (1-t)A + tB.$$

Theorem 2: *If A, B, and C are collinear, then there are real numbers x, y, and z not all 0 such that*

$$x + y + z = 0 \quad \textit{and} \quad xA + yB + zC = 0.$$

Theorem 3: *If A, B, and C are points and there are real numbers x, y, and z not all 0 such that*

$$x + y + z = 0 \quad \textit{and} \quad xA + yB + zC = 0,$$

then A, B, and C are collinear.

Theorem 4: *If A, B, and C are not collinear and if there are real numbers x, y, and z such that*

$$x + y + z = 0 \quad \textit{and} \quad xA + yB + zC = 0,$$

then $x = y = z = 0$.