MATH 532, 736I: MODERN GEOMETRY

Name ______________________________

Test #2 (1994)

Show All Work
Points: Problem (7) is worth 16 points; each of the other problems is worth 12 points.

(1) Theorems are listed on the last page of this test. They may or may not have the numbering that you are accustomed to them having from class. Prove Theorem 3 using Theorem 1 (but not Theorem 2 or Theorem 4).

(2) Let A, B, and C be 3 noncollinear points. Let D be the intersection of the (extended) altitudes of ΔABC. Let M_A be the midpoint of $\overline{BC}$, and let Q_A be the midpoint of $\overline{AD}$. Let $N = (A+B+C+D)/4$. Prove that the distance from N to M_A is the same as the distance for N to Q_A. This is part of the 9−point circle theorem, so you should not make use of the 9−point circle theorem in doing this problem.

(3) Let $A = (1, 2)$ and $B = (3, 2)$. Calculate each of the following, and put the answer in the appropriate box. Each answer should be either a specific number (write it down) or a specific point (write down its coordinates). The vector parts are meant to be direct computations. The translation and rotation parts can be done easily by interpreting the problems geometrically. (You can however choose to do these problems in a different manner).

(a) $2B - 3A = \boxed{}$

(b) $(B - A)^2 = \boxed{}$

(c) $T_A(B) = \boxed{}$

(d) $R_{\pi/2,A}(B) = \boxed{}$

(4) On the second to the last page is a drawing consisting of 2 triangles ΔABC and $\Delta A'B'C'$ which are perspective from the line passing through X, Y, and Z. Suppose we want to prove that these triangles are also perspective from a point. Let P be the point of intersection of the lines $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$. To show that $\overleftrightarrow{CC'}$ also passes through P, we can consider two triangles which we know to be perspective from a point. The vertices of these two triangles (not necessarily the edges) and the point they are perspective from are all in the drawing. Determine the two triangles and the point they are perspective from and indicate your answers in the boxes below. (No work necessary.)

Triangles $\boxed{}$ and $\boxed{}$ are perspective from point $\boxed{}$.

(Note: You can complete the boxes to make a sentence which is true, but that will not necessarily insure that you have a correct answer. A correct answer must correspond to establishing that $\overleftrightarrow{CC'}$ passes through P as discussed above.)

(5) The Pythagorean Theorem states that if ΔABC is a right triangle with legs (the shorter two sides) of length a and b and with hypotenuse (the longer side) of length c, then $a^2 + b^2 = c^2$. Prove the converse of the Pythagorean Theorem by making use of vectors. (Do NOT prove the Pythagorean Theorem!!) In other words, suppose ΔABC is a given triangle with side $\overline{AB}$ of length a, side $\overline{BC}$ of length b, and side $\overline{AC}$ of length c, and suppose that $a^2 + b^2 = c^2$. Using vectors, prove that $\angle ABC$ must be a right angle. (Comment and Hint: A proof that does not make appropriate use of vectors will be worth 0 points. A correct solution should involve interpreting the equation $a^2 + b^2 = c^2$ in terms of vectors and simplifying the result to deduce that $\angle ABC$ must be a right angle.)

(6) Let $P_1, P_2, \ldots, P_8$ be 8 (not necessarily distinct) points. Let $A_0 = A$ be an arbitrary point. For $j \in \{1, 2, \ldots, 8\}$, define A_j as the point you get by rotating A_{j-1} about P_j by π. Set $Q_1 = P_1$, $Q_2 = P_4$, $Q_3 = P_5$, $Q_4 = P_6$, $Q_5 = P_7$, $Q_6 = P_2$, $Q_7 = P_3$, and $Q_8 = P_8$. Let $B_0 = A$. For $j \in \{1, 2, \ldots, 8\}$, define B_j as the point you get by rotating B_{j-1} about Q_j by π. Prove that $A_8 = B_8$.

(7) Let A, B, and C be points, and let the function $f(x, y)$ be defined as follows. First f rotates each point (x, y) about the point A by $2\pi/3$, then it takes the result and rotates it about B by $4\pi/3$, and then it takes that result and rotates it about the point C by $2\pi/3$. Also, suppose $f(1, 1) = (5, 7)$. Determine whether f is a translation or a rotation and write it in the form T_D or $R_{\alpha,D}$ giving specific values for D or for α and D (i.e., if $f = T_D$, tell me what D is, and if $f = R_{\alpha,D}$, tell me what α and D are). You should make use of a theorem from class about the product of TWO rotations, but keep in mind that the theorem only deals with TWO rotations. Justify your steps.

(8) Let A, B, and C be 3 noncollinear points, and let A', B', and C' be 3 noncollinear points with $A \neq A'$, $B \neq B'$, and $C \neq C'$. Suppose that the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are parallel. Suppose further that $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ intersect at some point P, that $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ intersect at some point Q, and that $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ intersect at some point R. The next two pages contain a proof that P, Q, and R are collinear except there are some boxes which need to be filled. Complete the proof by filling in the boxes. There may be more than one correct way to fill in a box. The goal is to end up with a correct proof that P, Q, and R are collinear.

Proof: Since $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are parallel, there are real numbers k_1 and k_2 such that

$$A' - A = k_1(B' - B) = k_2(C' - C).$$

We get that

$$A - k_1 B = A' - k_1 B'.$$

We first explain why $k_1 \neq \boxed{}$.

Give explanation here: $\boxed{}$

Hence,

$$\left(\frac{1}{1-k_1}\right) A + \left(\frac{-k_1}{1-k_1}\right) B = \left(\frac{1}{1-k_1}\right) A' + \left(\frac{-k_1}{1-k_1}\right) B'.$$

By Theorem 1 (from the last page of this exam) with $t = \boxed{}$, we see that the expression on the left above is a point on line $\overleftrightarrow{AB}$ and that the expression on the right above is a point on line $\overleftrightarrow{A'B'}$. Therefore, we get that

(1) $$(1 - k_1)P = A - k_1 B.$$

Similarly, from $k_1 B - k_2 C = k_1 B' - k_2 C'$, we deduce that

(2) $\boxed{}$.

(Your answer should contain Q, B, and C.)

Also, from $k_2C - A = k_2C' - A'$, we deduce that

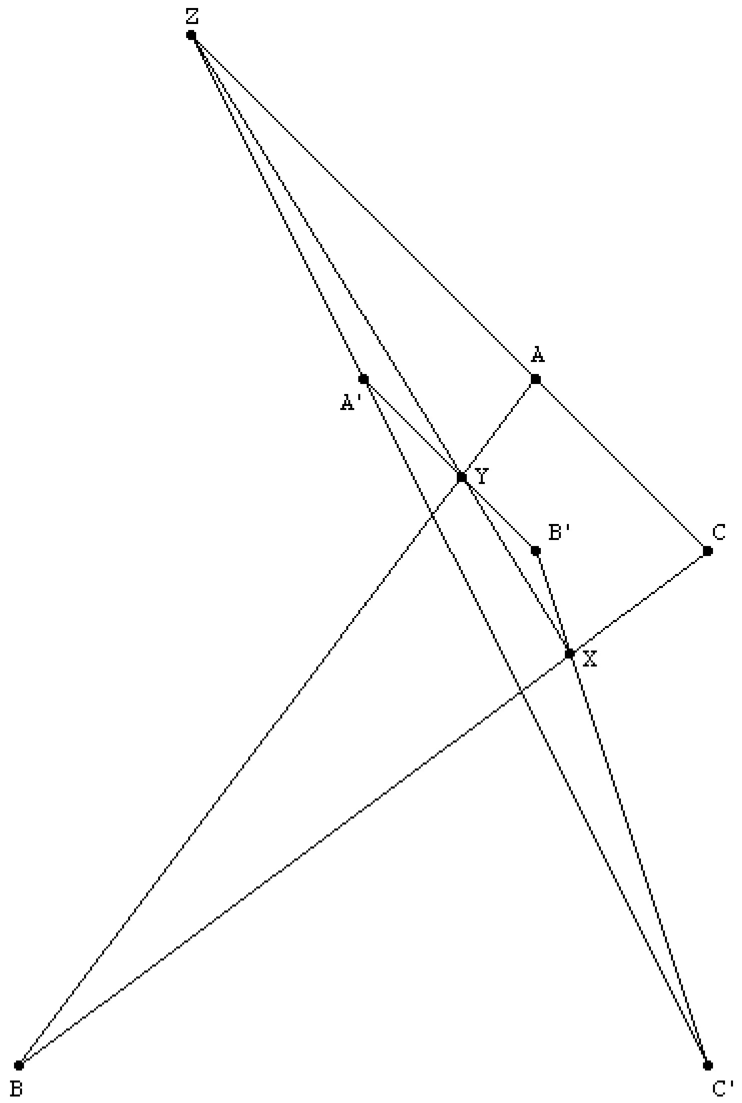
(3) $\boxed{}$.

(Your answer should contain R, A, and C.)

Therefore, from (1), (2), and (3),

$$(1 - k_1)P + (k_1 - k_2)Q + (k_2 - 1)R = \overrightarrow{0}.$$

The result follows from Theorem $\boxed{}$ (on the last page of this test).



Problem 4

INFORMATION PAGE

Theorem 1: *Let A and B be distinct points. Then C is a point on line $\overleftrightarrow{AB}$ if and only if there is a real number t such that*

$$C = (1-t)A + tB.$$

Theorem 2: *If A, B, and C are collinear, then there are real numbers x, y, and z not all 0 such that*

$$x + y + z = 0 \quad \textit{and} \quad xA + yB + zC = 0.$$

Theorem 3: *If A, B, and C are points and there are real numbers x, y, and z not all 0 such that*

$$x + y + z = 0 \quad \textit{and} \quad xA + yB + zC = 0,$$

then A, B, and C are collinear.

Theorem 4: *If A, B, and C are* not *collinear and if there are real numbers x, y, and z such that*

$$x + y + z = 0 \quad \textit{and} \quad xA + yB + zC = 0,$$

then $x = y = z = 0$.

$$T_{(a,b)} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_{\theta,(x_1,y_1)} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & x_1(1-\cos(\theta)) + y_1\sin(\theta) \\ \sin(\theta) & \cos(\theta) & -x_1\sin(\theta) + y_1(1-\cos(\theta)) \\ 0 & 0 & 1 \end{pmatrix}$$

$$T_{(a,b)} = R_{\pi,(a/2,b/2)} R_{\pi,(0,0)}$$