MATH 532, 736I: MODERN GEOMETRY

Name ______________________________________

Test #2 (1992)

Show All Work
Points: Part I (20 pts), Part II (38 pts), Part III (42 pts)

Part I. Each of the following is something or part of something you were asked to memorize. The problems in this section are worth 10 points each.

(1) Theorems are listed on the last page of this test. They may or may not have the numbering that you are accustomed to them having from class. Prove Theorem 2 using Theorem 1 (but not Theorem 3 or Theorem 4).

(2) Let A, B, and C be 3 noncollinear points. Let M_A be the midpoint of $\overline{BC}$ and let M_B be the midpoint of $\overline{AC}$. Let D be the intersection of the (extended) altitudes of ΔABC. Let $N = (A + B + C + D)/4$. Prove that the distance from N to M_A is the same as the distance for N to M_B. This is part of the $9-$point circle theorem, so you should not make use of the $9-$point circle theorem in doing this problem. (Comment: I have thrown in the parenthetical "extended" in front of "altitudes" because it is possible that the altitudes intersect at a point exterior to ΔABC.)

Part II. Each of the following is closely related to a homework problem. You should not refer to the homework problem done in class when doing these problems (in other words, show all of the necessary steps for the problems here). The first problem is worth 10 points and each of the other two are worth 14 points.

(1) The word 1010110 differs from a codeword by 1 digit. Use the parity check matrix discussed in class to determine what the codeword is.

(2) Let A, B, and C be 3 noncollinear points. Using vectors, show that the altitudes (when extended) for ΔABC are concurrent (i.e., show that there is a point which lies on all 3 altitudes).

(3) Let A be the point $(1, 3)$. The function $f(x, y)$ is defined as follows. First f rotates (x, y) about the point A by π, then it takes the result and translates it by A, and then it takes that result and rotates it about the point A by $\pi/2$. Explain why f is a rotation about some point B by $3\pi/2$ and compute the coordinates of the point B. Put point B in the form "(a, b)" in the box below. (Comment: For the first part of this problem, you should make use of a theorem from class about the product of TWO rotations, but keep in mind that the theorem only deals with TWO rotations. Justify your steps. You may want to use information on the Information Page at the end of the test.)

Point B is $\boxed{}$.

Part III. The first problem in this section is worth 14 points, the second problem is worth 12 points, and the last problem is worth 16 points.

(1) Let ℓ and m be parallel lines. Let A and B be distinct points on ℓ, and let C and D be distinct points on m. Let E be the midpoint of $\overline{AC}$ and let F be the midpoint of $\overline{BD}$, and suppose that $E \neq F$. Using vectors, prove that $\overleftrightarrow{EF}$ is parallel to ℓ and m.

(2) Let P_1, P_2, P_3, and P_4 be 4 (not necessarily distinct) points. Let A be an arbitrary point. Beginning with $A_0 = A$, for $j \in \{1, 2, 3, 4\}$, define A_j as the point you get by rotating A_{j-1} about P_j by π. Set $Q_1 = P_3$, $Q_2 = P_4$, $Q_3 = P_1$, and $Q_4 = P_2$. Beginning with $B_0 = A$, for $j \in \{1, 2, 3, 4\}$, define B_j as the point you get by rotating B_{j-1} about Q_j by π. Prove that $A_4 = B_4$. (See the first picture on the second from the last page for an example.)

(3) Let A, B, and C be 3 noncollinear points, and let A', B', and C' be 3 noncollinear points. Suppose that ΔABC and $\Delta A'B'C'$ are perspective from a point X in the plane. Suppose further that $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ intersect at some point P, that $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ intersect at some point Q, and that $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ are parallel. (See the second on the second from the last page.) The next two pages contain a proof that $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AC}$ are parallel except there are some boxes which need to be filled. Complete the proof by filling in the boxes. There may be more than one correct way to fill in a box. The goal is to end up with a correct proof that $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AC}$ are parallel.

Proof: By Theorem $\boxed{\quad}$ (from the Information Page at the end of this test), there are real numbers k_1, k_2, and k_3 such that

$$X = (1 - k_1)A + k_1 A' = (1 - k_2)B + k_2 B' = (1 - k_3)C + k_3 C'.$$

Next, we show that $k_1 \neq k_2$. Assume $k_1 = k_2$. Observe that $k_1 \neq \boxed{\quad}$ (answer here either 0 or 1) since otherwise we would have $X = A = B$, contradicting that A and B are distinct points. Also, $k_1 \neq \boxed{\quad}$ (answer here either 0 or 1) since otherwise we would have $\boxed{\qquad\qquad}$, contradicting that $\boxed{\quad}$ and $\boxed{\quad}$ are distinct points. We get that

$$(1 - k_1)A - (1 - k_2)B = k_2 B' - k_1 A'$$

and that the vectors $\boxed{\qquad\qquad}$ and $\boxed{\qquad\qquad}$ either have the same direction or the exact opposite direction. This contradicts that the point $\boxed{\quad}$ exists. Hence, $k_1 \neq k_2$. Thus,

$$\frac{1 - k_1}{k_2 - k_1}A + \frac{k_2 - 1}{k_2 - k_1}B = \frac{k_2}{k_2 - k_1}B' + \frac{-k_1}{k_2 - k_1}A'.$$

By Theorem 1 with $t = \boxed{\qquad\qquad}$, we see that the expression on the left above is a point on line $\overleftrightarrow{AB}$. By Theorem 1 with $t = \boxed{\qquad\qquad}$, we see that the expression on the right above is a point on line $\overleftrightarrow{A'B'}$. Therefore, we get that

$$P = \frac{1 - k_1}{k_2 - k_1}A + \frac{k_2 - 1}{k_2 - k_1}B.$$

Hence,

$$(k_2 - k_1)P = (1 - k_1)A + (k_2 - 1)B. \tag{1}$$

Using that

$$(1 - k_2)B - (1 - k_3)C = k_3C' - k_2B',$$

we similarly obtain that $k_2 \neq k_3$, that

$$\frac{1 - k_2}{k_3 - k_2}B + \frac{k_3 - 1}{k_3 - k_2}C = \frac{k_3}{k_3 - k_2}C' + \frac{-k_2}{k_3 - k_2}B',$$

and that

$$(k_3 - k_2)Q = (1 - k_2)B + (k_3 - 1)C. \tag{2}$$

From

$$(1 - k_3)C - (1 - k_1)A = k_1A' - k_3C',$$

we similarly obtain that either

$$\boxed{\qquad\qquad\qquad\qquad\qquad\qquad} \tag{3}$$

or

$$\frac{1 - k_3}{k_1 - k_3}C + \frac{k_1 - 1}{k_1 - k_3}A = \frac{k_1}{k_1 - k_3}A' + \frac{-k_3}{k_1 - k_3}C'. \tag{4}$$

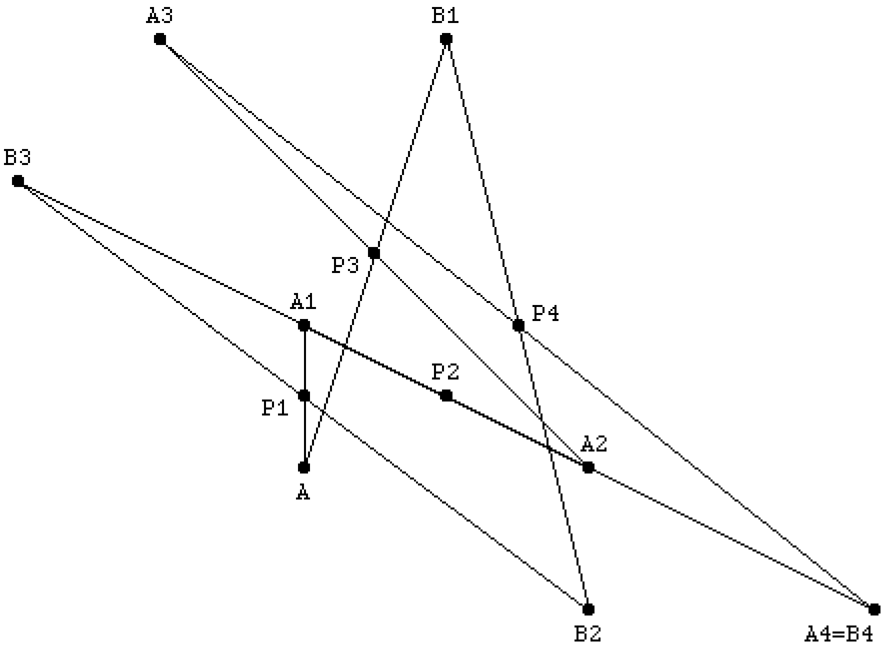
If (4) holds, then we could deduce that there is a point on both line $\boxed{\qquad\qquad}$ and line $\boxed{\qquad\qquad}$, giving a contradiction. Thus, (3) must hold. We get from (1) and (2) that

$$(k_2 - k_1)P + (k_3 - k_2)Q = (1 - k_1)A + (k_3 - 1)C$$

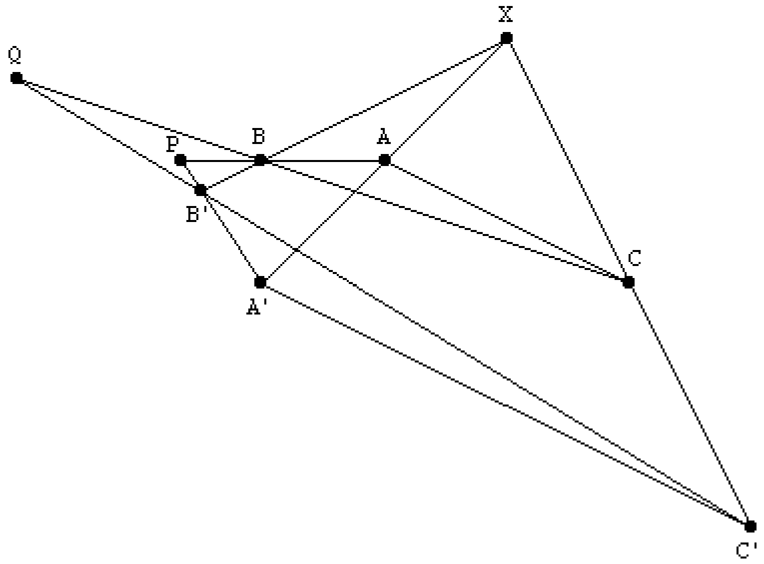
so that

$$(k_2 - k_1)(P - Q) = \boxed{}.$$

Observe that $P \neq Q$ since otherwise we would have that the points A, B, and $\boxed{}$ are collinear, which isn't the case. Since $k_1 \neq k_2$, we obtain that the lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AC}$ are parallel, completing the proof. ■



Part III, Problem 2



Part III, Problem 3

INFORMATION PAGE

Theorem 1: *Let A and B be distinct points. Then C is a point on line $\overleftrightarrow{AB}$ if and only if there is a real number t such that*
$$C = (1-t)A + tB.$$

Theorem 2: *If A, B, and C are points and there are real numbers x, y, and z not all 0 such that*
$$x + y + z = 0 \quad \textit{and} \quad xA + yB + zC = 0,$$
then A, B, and C are collinear.

Theorem 3: *If A, B, and C are collinear, then there are real numbers x, y, and z not all 0 such that*
$$x + y + z = 0 \quad \textit{and} \quad xA + yB + zC = 0.$$

Theorem 4: *If A, B, and C are not collinear and if there are real numbers x, y, and z such that*
$$x + y + z = 0 \quad \textit{and} \quad xA + yB + zC = 0,$$
then $x = y = z = 0$.

$$T_{(a,b)} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_{\theta,(x_1,y_1)} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & x_1(1-\cos(\theta)) + y_1\sin(\theta) \\ \sin(\theta) & \cos(\theta) & -x_1\sin(\theta) + y_1(1-\cos(\theta)) \\ 0 & 0 & 1 \end{pmatrix}$$

$$T_{(a,b)} = R_{\pi,(a/2,b/2)}R_{\pi,(0,0)}$$