MATH 532, 736I: MODERN GEOMETRY

Test 2 Solutions

Test #2 (2011)

1) Theorems are listed on the last page of this test. They may or may not have the numbering that you are accustomed to them having from class. Prove Theorem 2 using Theorem 1 (but not Theorem 3 or Theorem 4)

   If $A = B$, then take $x = 1$, $y = -1$, and $z = 0$.

   Suppose now that $A \neq B$. By Theorem 1 there is a real number t such that $C = (1-t)A + tB$. Let $x = 1 - t$, $y = t$, and $z = -1$. Then x, y, and z are not all 0, $x + y + z = 0$, and $xA + yB + zC = \vec{0}$.

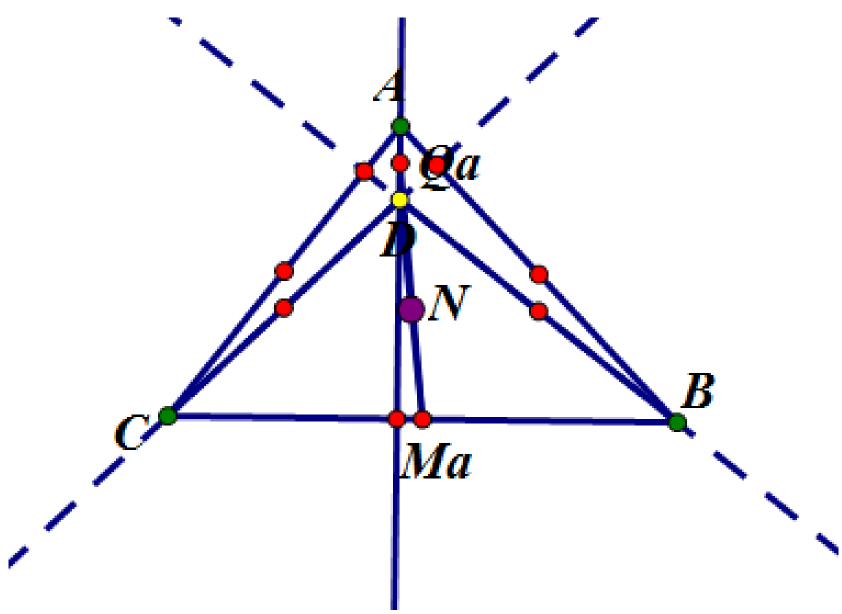
2) Let A, B, and C be 3 noncollinear point. Let M_a be the midpoint of $\overline{BC}$ and let D be the intersection of the (extended) altitudes of $\triangle ABC$. Let Q_a be the midpoint of $\overline{AD}$. Finally, let $N = (A+B+C+D)/4$. Prove that the distance from N to M_a is the same the distance from N to Q_a. This is part of the 9-point circle theorem, so you should not make use of the 9-point circle theorem in doing this problem.

   Show that $|NM_A| = |NQ_A|$ by showing that N is the midpoint of $\overline{M_A Q_A}$.

   $M_A = \frac{B+C}{2}$ and $Q_A = \frac{A+D}{2}$

   $\frac{M_A + Q_A}{2} = \frac{(B+C+A+D)/2}{2} = \frac{A+B+C+D}{4} = N$

   Hence, N is the midpoint of $\overline{M_A Q_A}$ and $|\overline{NM_A}| = |\overline{NQ_A}|$

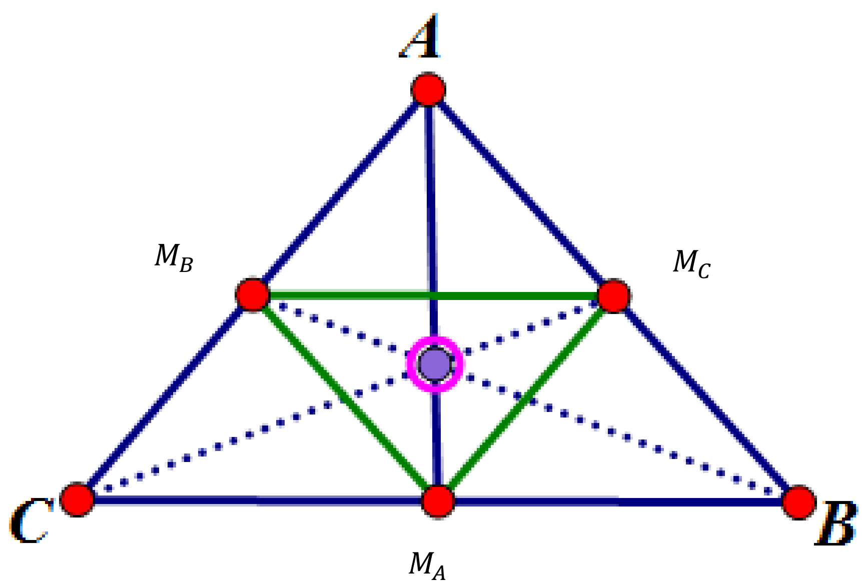


3) The centroid of a triangle is the point that is the average of its vertices. In other words, the point *(U + V + W)/ 3* is the centroid of $\triangle UVW$. For a $\triangle ABC$, let M_A be the midpoint of side $\overline{BC}$, let M_B be the midpoint of side $\overline{AC}$, and let M_C be the midpoint of side $\overline{AB}$. Show that the centroid of $\triangle M_A M_B M_C$ is equal to the centroid of $\triangle ABC$.

Show that $\frac{M_A+M_B+M_C}{3} = \frac{A+B+C}{3}$

$M_A = \frac{B+C}{2}$, $M_B = \frac{A+C}{2}$, and $M_C = \frac{A+B}{2}$

The centroid for $\triangle M_A M_B M_C = \frac{M_A+M_B+M_C}{3} = \frac{(B+C+A+C+A+B)/2}{3} = \frac{2(A+B+C)}{6} = \frac{A+B+C}{3}$



4) For each part below, the function $f(x,y)$ is defined as follows. First f rotates (x,y) about the point $A=(-1,1)$ by π and then it takes the result and translates it by the point $B=(-2,3)$ and then it rotates this result about the point $C=(-2,-1)$ by $\frac{\pi}{2}$. Thus, we can view f as being $R_{\pi/2,C}T_B R_{\pi,A}$. As usual, all rotations are counter-clockwise.

From the translation and rotation matrices we obtain

$$R_{\pi/2,C}=\begin{pmatrix}0&-1&-3\\1&0&1\\0&0&1\end{pmatrix},\quad T_B=\begin{pmatrix}1&0&-2\\0&1&3\\0&0&1\end{pmatrix},\text{ and } R_{\pi,A}=\begin{pmatrix}-1&0&-2\\0&-1&2\\0&0&1\end{pmatrix}$$

Multiply the first two matrices to find $\begin{pmatrix}0&-1&-3\\1&0&1\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&-2\\0&1&3\\0&0&1\end{pmatrix}=\begin{pmatrix}0&-1&-6\\1&0&-1\\0&0&1\end{pmatrix}$

Multiply that result by the final matrix $\begin{pmatrix}0&-1&-6\\1&0&-1\\0&0&1\end{pmatrix}\begin{pmatrix}-1&0&-2\\0&-1&2\\0&0&1\end{pmatrix}=\begin{pmatrix}0&1&-8\\-1&0&-3\\0&0&1\end{pmatrix}$

a) Calculate $f(4,1)$

$$f(4,1)=\begin{pmatrix}0&1&-8\\-1&0&-3\\0&0&1\end{pmatrix}\begin{pmatrix}4\\1\\1\end{pmatrix}=\begin{pmatrix}-7\\-7\\1\end{pmatrix}$$

Answer: (-7,-7)

b) Find a point (x_0,y_0) satisfying $(x_0,y_0)=(x_0,y_0)$.

(x_0,y_0): $\left(\frac{-11}{2},\frac{5}{2}\right)$

$$\begin{pmatrix}0&1&-8\\-1&0&-3\\0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\1\end{pmatrix}=\begin{pmatrix}y-8\\-x-3\\1\end{pmatrix}$$

$$\begin{matrix}x=y-8\\y=-x-3\end{matrix}\Rightarrow\begin{matrix}x-y=-8\\x+y=-3\\\hline 2x=-11\end{matrix}$$

We use simultaneous equations and substitution to find the value of x and y.

From here we find $x=\frac{-11}{2}$ and $y=-\left(\frac{-11}{2}\right)-3\Rightarrow y=\frac{5}{2}$

c) Determine whether f is a translation or a rotation. If f is a translation, express f in the form $T_{(a,b)}$ where *a* and *b* are explicit numbers. If f is a rotation, express f in the form $R_{\theta,(a,b)}$ where $\theta, a,$ and b are explicit numbers.

$f = R_{\pi/2,C} T_B R_{\pi,A}$

Using Theorem and the fact that a translation is the sum of two rotations, we can rewrite as follows.

$$f = R_{\pi/2,C} R_{\pi,*} R_{\pi,,*} R_{\pi,A}$$

$$\left(R_{\pi/2,C} R_{\pi,*} R_{\pi,,*}\right) R_{\pi,A}$$

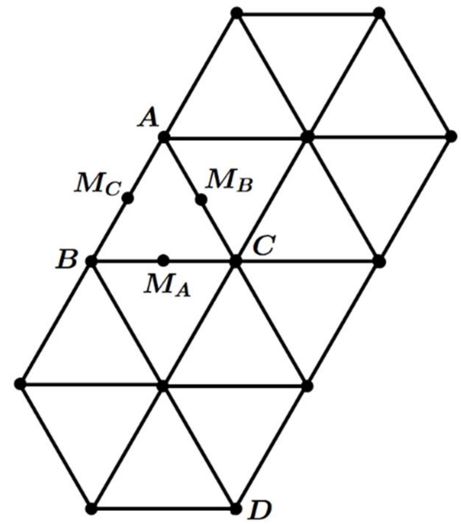
$$R_{5\pi/2,*} R_{\pi,A}$$

$$R_{7\pi/2,*} \Rightarrow R_{3\pi/2,*}$$

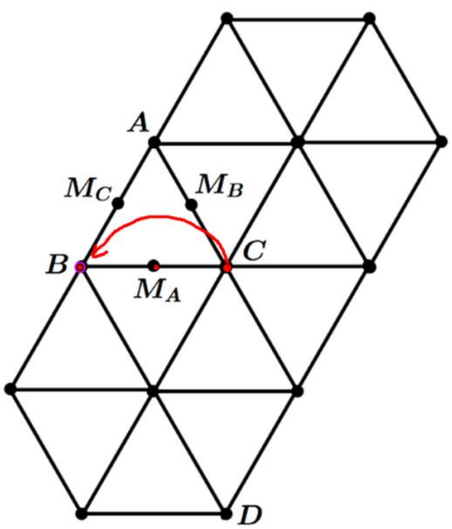
Since $3\pi/2$ is not a multiple of 2π, this is a rotation. From (b), we know the point about which the rotation occurs must be $\left(\frac{-11}{2}, \frac{5}{2}\right)$ since $f(x_0, y_0) = (x_0, y_0)$

f: $R_{3\pi/2,\left(\frac{-11}{2},\frac{5}{2}\right)}$

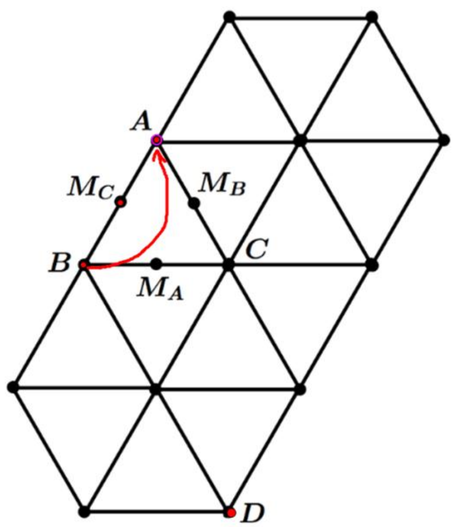
5) The Picture to the right shows 14 congruent equilateral triangles. One of these triangles is $\triangle ABC$. The point M_A is the midpoint of segment $\overline{BC}$, the point M_B is the midpoint of segment $\overline{AC}$, and M_C is the midpoint of segment $\overline{AB}$. Consider the function f that is a rotation about M_A by π, followed by a rotation about M_C by π, and then followed by a rotation about M_B by π. So $f = R_{\pi,M_B}R_{\pi,M_C}R_{\pi,M_A}$



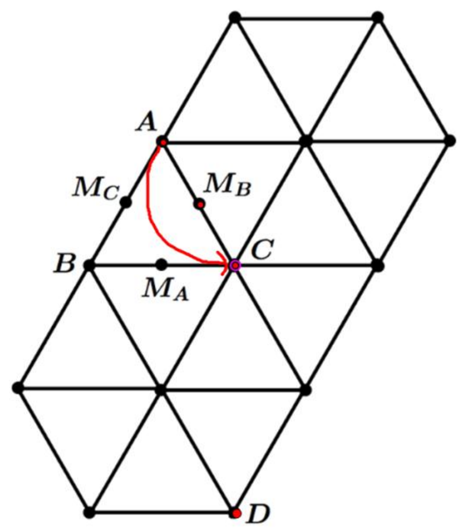
a. What point is $f(C)$? Since we can simply use the principle of rotation and visualize of how C moves, we can determine that $f(C) = C$



R_{π,M_A}



R_{π,M_C}

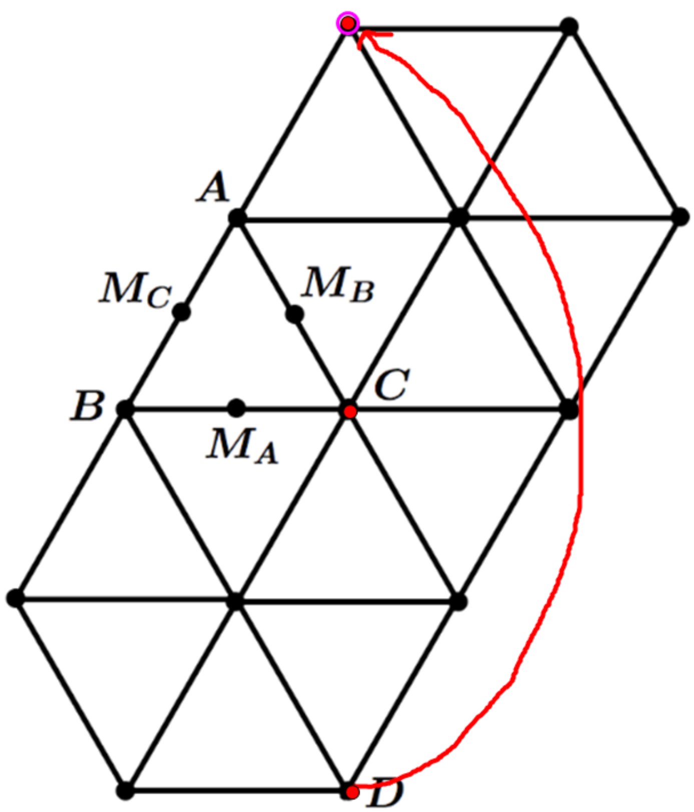


R_{π,M_B}

b) What point is $f(D)$? Circle the point to the right and justify your answer by using part (a) and Theorem 5 from the last page of the test.

IMPORTANT: You must explain your answer by using the theorem even if you have another reason for your answer. I want to know if you understand how the theorem gives the answer.

Using Theorem 5, we know that f is a rotation about some point by π. Since the sum of the rotations is equal to 3π and maps point C to itself, C is the point about which f rotates and $f = R_{\pi,C}$. Hence $f(D)$ is rotating $f(D)$ about point $C\ by\ \pi$, and that is the point which is circled.



6) This is the same problem as Problem 3 Part III from Test 2 of 1992. See the solutions to that test for the solution to this problem.