MATH 532, 736I: MODERN GEOMETRY

Test 2, Spring 2012 Name ______________________________

Show All Work

Instructions: This test consists of 4 pages (one is an information page). Put your name at the top of this page and at the top of the first page of the packet of blank paper given to you. Work each problem in the packet of paper unless it is indicated that you can or are to do the work below. Fill in the boxes below with your answers. Show ALL of your work. Do NOT use a calculator.

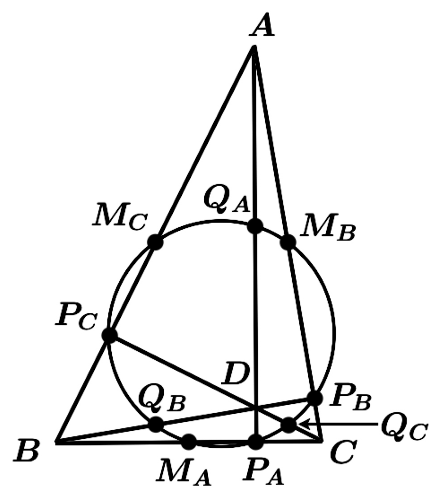
10 pts (1) If you are told that A, B and C are points with $\angle ABC$ a right angle, how would you express this information using the notation we have been using in class? (For example, a wrong answer would be $(C - A)(B + C) = 1$, but a correct answer would look somewhat similar.)

Answer: [ ]

10 pts (2) Let $A = (2, 8)$ and $B = (-2012, 2012)$. If f is a rotation by π about the point C and $f(A) = B$, then what is the value of C? Simplify your answer. (It's a point, so I am looking for an answer of the form (x_0, y_0) where x_0 and y_0 are specific numbers.)

Answer: [ ]

16 pts (3) Let A, B, and C be 3 noncollinear points. Let M_A be the midpoint of $\overline{BC}$, and let M_B be the midpoint of $\overline{AC}$. Let D be the intersection of the (extended) altitudes of ΔABC. Let $N = (A + B + C + D)/4$. Prove that the distance from N to M_A is the same as the distance from N to M_B. This is part of the 9-point circle theorem, so you should not make use of the 9-point circle theorem in doing this problem. Instead you want to prove this part of the theorem.



24 pts (4) Let A, B, and C be 3 noncollinear points, and let A', B', and C' be 3 noncollinear points with $A \neq A'$, $B \neq B'$, and $C \neq C'$. Suppose that the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are parallel. Suppose there are points P, Q and R in the (finite) plane such that $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ intersect at some point P, that $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ intersect at some point Q, and that $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ intersect at some point R. See the last page for a picture. The next page contains a proof that P, Q, and R are collinear except there are some boxes which need to be filled. Complete the proof by filling in the boxes. There may be more than one correct way to fill in a box. The goal is to end up with a correct proof that P, Q, and R are collinear.

Proof: Since $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are parallel, there are real numbers k_1 and k_2 such that

$$A' - A = k_1(B' - B) = k_2(C' - C).$$

We get that

$$A - k_1 B = A' - k_1 B'.$$

We first explain why $k_1 \neq \boxed{\quad}$.

Give explanation here: $\boxed{\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad}$

Hence,

$$\left(\frac{1}{1-k_1}\right) A + \left(\frac{-k_1}{1-k_1}\right) B = \left(\frac{1}{1-k_1}\right) A' + \left(\frac{-k_1}{1-k_1}\right) B'.$$

By Theorem 1 (from the last page of this exam) with $t = \boxed{\quad\quad}$, we see that the expression on the left above is a point on line $\overleftrightarrow{AB}$ and that the expression on the right above is a point on line $\overleftrightarrow{A'B'}$. Therefore, we get that

(1) $$(1 - k_1)P = A - k_1 B.$$

Similarly, from $k_1 B - k_2 C = k_1 B' - k_2 C'$, we deduce that

(2) $$\boxed{\quad\quad\quad\quad\quad\quad\quad\quad}.$$

(Your answer should contain Q, B, and C.)

Also, from $k_2 C - A = k_2 C' - A'$, we deduce that

(3) $$\boxed{\quad\quad\quad\quad\quad\quad\quad\quad}.$$

(Your answer should contain R, A, and C.)

Therefore, from (1), (2), and (3),

$$(1 - k_1)P + (k_1 - k_2)Q + (k_2 - 1)R = \overrightarrow{0}.$$

The result follows from Theorem $\boxed{\quad}$ given on the last page of this test (use the numbering of the theorems given there).

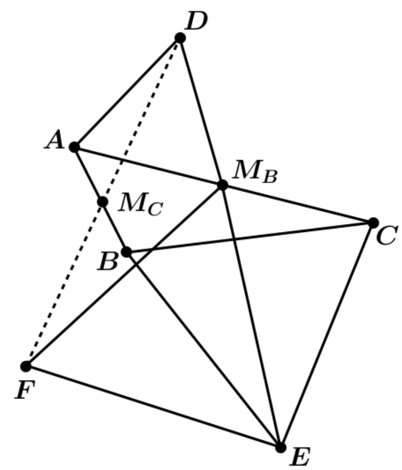
16 pts

(5) The function $f(x, y)$ is defined as follows. First f rotates (x, y) about the point $(2, -3)$ by $\pi/2$, then it takes the result and translates it by $(20, 12)$, and then it takes that result and rotates it about the point $(-1, 6)$ by $\pi/2$. Decide whether f is a translation or a rotation. If f is a translation, express f in the form $T_{(a,b)}$ where a and b are explicit numbers. If f is a rotation, express f in the form $R_{\phi,(a,b)}$ where ϕ, a, and b are explicit numbers. Be sure to show work.

$f = \boxed{}$ (either $T_{(a,b)}$ or $R_{\phi,(a,b)}$ but with specific numbers for a, b, and possibly ϕ)

24 pts

(6) Let M_B be the midpoint of side $\overline{AC}$ of a triangle $\triangle ABC$, and let M_C be the midpoint of side $\overline{AB}$. Let D, E and F be as shown in the picture with each of the triangles $\triangle DAM_B$, $\triangle BEC$ and $\triangle M_BFE$ equilateral. The purpose of this problem is to prove the points F, M_C and D are collinear and M_C is the midpoint of $\overline{FD}$.



(a) Let $f = R_{\pi,M_C} R_{\pi/3,E} R_{\pi,M_B}$. What is the value of $f(A)$?

$f(A) = \boxed{}$

(b) With f as in part (a), we know from an important theorem in class that $f = R_{\pi/3,P}$ for some point P. What is the point P? (It is one of the points in the picture.)

$P = \boxed{}$

(c) Using part (b), what is the value of $f(M_B)$?

$f(M_B) = \boxed{}$

(d) What is the value of $R_{\pi/3,E}\big(R_{\pi,M_B}(M_B)\big)$? (In other words, what do the first two rotations in the definition of f do to the point M_B ?)

$R_{\pi/3,E}\big(R_{\pi,M_B}(M_B)\big) = \boxed{}$

(e) Using parts (c) and (d), what is $R_{\pi,M_C}(F)$? Explain how you are using parts (c) and (d) to get your answer to this part in the blank paper provided with the test. This explanation is the main part of the problem.

$R_{\pi,M_C}(F) = \boxed{}$ (Don't forget to justify your answer on the blank paper.)

(f) On the blank paper provided with the test, explain why the points F, M_C and D are collinear and M_C is the midpoint of $\overline{FD}$. (This can be brief, just one or two sentences.)

INFORMATION PAGE

Theorem 1: *Let A and B be distinct points. Then C is a point on line $\overleftrightarrow{AB}$ if and only if there is a real number t such that*

$$C = (1-t)A + tB.$$

Theorem 2: *If A, B and C are points and there are real numbers x, y, and z not all 0 such that*

$$x + y + z = 0 \quad \text{and} \quad xA + yB + zC = \overrightarrow{0},$$

then A, B and C are collinear.

Theorem 3: *If A, B and C are collinear, then there are real numbers x, y and z not all 0 such that*

$$x + y + z = 0 \quad \text{and} \quad xA + yB + zC = \overrightarrow{0}.$$

Theorem 4: *If A, B and C are not collinear and if there are real numbers x, y and z such that*

$$x + y + z = 0 \quad \text{and} \quad xA + yB + zC = \overrightarrow{0},$$

then $x = y = z = 0$.

Theorem 5: *Let $\alpha_1, \ldots, \alpha_n$ be real numbers (not necessarily distinct), and let $A_1, \ldots, A_n$ and $B_1, \ldots, B_k$ be points (not necessarily distinct). Let f be a product of the n rotations R_{α_j, A_j} and the k translations T_{B_j} with each of the n rotations and k translations occurring exactly once in the product. If $\alpha_1 + \cdots + \alpha_n$ is not an integer multiple of 2π, then there is point C such that*

$$f = R_{\alpha_1+\alpha_2+\cdots+\alpha_n, C}.$$

If $\alpha_1 + \cdots + \alpha_n$ is an integer multiple of 2π, then f is a translation.

$$T_{(a,b)} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \qquad T_{(a,b)} = R_{\pi,(a/2,b/2)} R_{\pi,(0,0)}$$

$$R_{\theta,(x_1,y_1)} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & x_1(1-\cos(\theta)) + y_1\sin(\theta) \\ \sin(\theta) & \cos(\theta) & -x_1\sin(\theta) + y_1(1-\cos(\theta)) \\ 0 & 0 & 1 \end{pmatrix}$$

Picture for Problem 4: