MATH 532, 736I: MODERN GEOMETRY

Test #2 (2011) Name ______________________

Show All Work

Instructions: There are 100 points possible on the test. The value of each problem appears to the left of each problem number. If there are boxes with these test questions, fill them in appropriately with your answers. Note that there is important information on the last page of the test.

12 pts (1) Theorems are listed on the last page of this test. They may or may not have the numbering that you are accustomed to them having from class. Prove Theorem 2 using Theorem 1 (but not Theorem 3 or Theorem 4).

12 pts (2) Let A, B, and C be 3 noncollinear points. Let M_A be the midpoint of $\overline{BC}$ and let D be the intersection of the (extended) altitudes of $\triangle ABC$. Let Q_A be the midpoint of $\overline{AD}$. Finally, let $N = (A+B+C+D)/4$. Prove that the distance from N to M_A is the same as the distance for N to Q_A. This is part of the 9-point circle theorem, so you should not make use of the 9-point circle theorem in doing this problem.

16 pts (3) The *centroid* of a triangle is the point that is the average of its vertices. In other words, the point $(U + V + W)/3$ is the centroid of $\triangle UVW$. For a $\triangle ABC$, let M_A be the midpoint of side $\overline{BC}$, let M_B be the midpoint of side $\overline{AC}$, and let M_C be the midpoint of side $\overline{AB}$. Show that the centroid of $\triangle M_AM_BM_C$ is equal to the centroid of $\triangle ABC$.

20 pts (4) For each part below, the function $f(x, y)$ is defined as follows. First f rotates (x, y) about the point $A = (-1, 1)$ by π and then it takes the result and translates it by the point $B = (-2, 3)$ and then it rotates this result about the point $C = (-2, -1)$ by $\pi/2$. Thus, we can view f as being $R_{\pi/2,C}T_BR_{\pi,A}$. As usual, all rotations are counter-clockwise.

(a) Calculate $f(4, 1)$.

Answer: $\boxed{}$

(b) Find a point (x_0, y_0) satisfying $f(x_0, y_0) = (x_0, y_0)$.

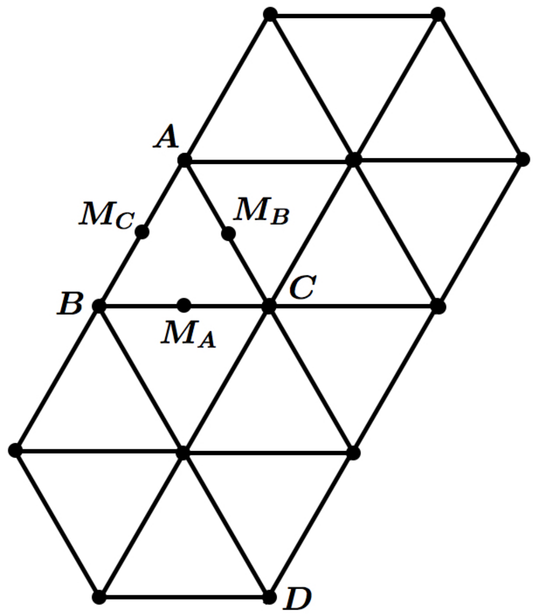
(x_0, y_0): $\boxed{}$

(c) Decide whether f is a translation or a rotation. If f is a translation, express f in the form $T_{(a,b)}$ where a and b are explicit numbers. If f is a rotation, express f in the form $R_{\phi,(a,b)}$ where ϕ, a, and b are explicit numbers.

f: $\boxed{}$

16 pts (5) The picture to the right shows 14 congruent equilateral triangles. One of these triangles is $\triangle ABC$. The point M_A is the midpoint of segment $\overline{BC}$, the point M_B is the midpoint of segment $\overline{AC}$, and the point M_C is the midpoint of segment $\overline{AB}$. Consider the function f that is a rotation about M_A by π, followed by a rotation about M_C by π and then followed by a rotation about M_B by π. So

$$f = R_{\pi,M_B} R_{\pi,M_C} R_{\pi,M_A}.$$



(a) What point is $f(C)$?

(b) What point is $f(D)$? Circle the point to the right and justify your answer by using part (a) and Theorem 5 from the last page of the test. **<u>IMPORTANT</u>:** You must explain your answer by using this theorem even if you have another reason for your answer. I want to know if you understand how the theorem gives the answer.

24 pts (6) Let A, B, and C be 3 noncollinear points, and let A', B', and C' be 3 noncollinear points. Suppose that ΔABC and $\Delta A'B'C'$ are perspective from a point X in the plane. Suppose further that $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ intersect at some point P, that $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ intersect at some point Q, and that $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ are parallel. The next two pages contain a proof that $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AC}$ are parallel except there are some boxes which need to be filled. Complete the proof by filling in the boxes. There may be more than one correct way to fill in a box. The goal is to end up with a correct proof that $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AC}$ are parallel. (There is a small picture at the bottom of the last page.)

Proof: By Theorem $\boxed{}$ (from the Information Page at the end of this test), there are real numbers k_1, k_2, and k_3 such that

$$X = (1-k_1)A + k_1A' = (1-k_2)B + k_2B' = (1-k_3)C + k_3C'.$$

Next, we show that $k_1 \neq k_2$. Assume $k_1 = k_2$. Observe that $k_1 \neq \boxed{}$ (answer here either 0 or 1) since otherwise we would have $X = A = B$, contradicting that A and B are distinct points. Also, $k_1 \neq \boxed{}$ (answer here either 0 or 1) since otherwise we would have $\boxed{}$, contradicting that $\boxed{}$ and $\boxed{}$ are distinct points. We get that

$$(1-k_1)A - (1-k_2)B = k_2B' - k_1A'$$

and that the vectors $\boxed{}$ and $\boxed{}$ either have the same direction or the exact opposite direction. This contradicts that the point $\boxed{\quad}$ exists. Hence, $k_1 \neq k_2$. Thus,

$$\frac{1-k_1}{k_2-k_1}A + \frac{k_2-1}{k_2-k_1}B = \frac{k_2}{k_2-k_1}B' + \frac{-k_1}{k_2-k_1}A'.$$

By Theorem 1 with $t = \boxed{\qquad}$, we see that the expression on the left above is a point on line $\overleftrightarrow{AB}$. By Theorem 1 with $t = \boxed{\qquad}$, we see that the expression on the right above is a point on line $\overleftrightarrow{A'B'}$. Therefore, we get that

$$P = \frac{1-k_1}{k_2-k_1}A + \frac{k_2-1}{k_2-k_1}B.$$

Hence,

(1) $$(k_2-k_1)P = (1-k_1)A + (k_2-1)B.$$

Using that

$$(1-k_2)B - (1-k_3)C = k_3C' - k_2B',$$

we similarly obtain that $k_2 \neq k_3$, that

$$\frac{1-k_2}{k_3-k_2}B + \frac{k_3-1}{k_3-k_2}C = \frac{k_3}{k_3-k_2}C' + \frac{-k_2}{k_3-k_2}B',$$

and that

(2) $$(k_3-k_2)Q = (1-k_2)B + (k_3-1)C.$$

From

$$(1-k_3)C - (1-k_1)A = k_1A' - k_3C',$$

we similarly obtain that either

(3) $$\boxed{\qquad\qquad\qquad\qquad}$$

or

(4) $$\frac{1-k_3}{k_1-k_3}C + \frac{k_1-1}{k_1-k_3}A = \frac{k_1}{k_1-k_3}A' + \frac{-k_3}{k_1-k_3}C'.$$

If (4) holds, then we could deduce that there is a point on both line $\boxed{\qquad}$ and line $\boxed{\qquad}$, giving a contradiction. Thus, (3) must hold. We get from (1) and (2) that

$$(k_2-k_1)P + (k_3-k_2)Q = (1-k_1)A + (k_3-1)C$$

so that

$$(k_2-k_1)(P-Q) = \boxed{\qquad\qquad\qquad\qquad}.$$

Observe that $P \neq Q$ since otherwise we would have that the points A, B, and $\boxed{\quad}$ are collinear, which isn't the case. Since $k_1 \neq k_2$, we obtain that the lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AC}$ are parallel, completing the proof. ■

INFORMATION FOR TEST

Theorem 1: *Let A and B be distinct points. Then C is a point on line $\overleftrightarrow{AB}$ if and only if there is a real number t such that*

$$C = (1-t)A + tB.$$

Theorem 2: *If A, B, and C are collinear, then there are real numbers x, y, and z not all 0 such that*

$$x + y + z = 0 \quad \text{and} \quad xA + yB + zC = \overrightarrow{0}.$$

Theorem 3: *If A, B, and C are points and there are real numbers x, y, and z not all 0 such that*

$$x + y + z = 0 \quad \text{and} \quad xA + yB + zC = \overrightarrow{0},$$

then A, B, and C are collinear.

Theorem 4: *If A, B, and C are not collinear and if there are real numbers x, y, and z such that*

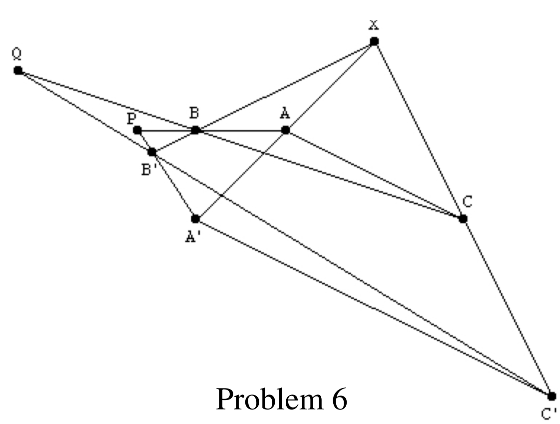
$$x + y + z = 0 \quad \text{and} \quad xA + yB + zC = \overrightarrow{0},$$

then $x = y = z = 0$.

Theorem 5: *Let α and β be real numbers (not necessarily distinct), and let A and B be points (not necessarily distinct). If $\alpha + \beta$ is not an integer multiple of 2π, then there is a point C such that $R_{\beta,B}R_{\alpha,A} = R_{\alpha+\beta,C}$. If $\alpha + \beta$ is an integer multiple of 2π, then $R_{\beta,B}R_{\alpha,A}$ is a translation.*

$$T_{(a,b)} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \qquad T_{(a,b)} = R_{\pi,(a/2,b/2)}R_{\pi,(0,0)}$$

$$R_{\theta,(x_1,y_1)} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & x_1(1-\cos(\theta)) + y_1\sin(\theta) \\ \sin(\theta) & \cos(\theta) & -x_1\sin(\theta) + y_1(1-\cos(\theta)) \\ 0 & 0 & 1 \end{pmatrix}$$



Problem 6