Math 532, 736I: Modern Geometry

Test 1, Spring 2013 (Solutions): Provided by Jeff Collins and Anil Patel

Part 1:

1. Axioms for a finite AFFINE plane of order n.

AA1: There exist at least 4 points, no 3 of which are collinear.

AA2: There exists at least 1 line with exactly n points on it.

AA3: Given any 2 distinct points, there is exactly 1 line that passes through these points.

AA4: Given any line l , and a point P not on l , there is exactly 1 line that does not intersect l and passes through P.

2. Model of a finite PROJECTIVE plane of order 3 (see appendix)

3. Two points have been circled in a 7 x 7 array of points. Using the model for a finite affine plane of order 7, finish circling the points that belong to the same line as given points (see appendix)

4. Consider the points (10,41) and (30,26) in a 59 x 59 array of points for our model of a finite affine plane of order 59. Find the equation of the line passing through these two points. Put your answer in the form y ≡ mx + k (mod 59) where m and k are among the numbers 0,1,2,3,…,58.

(10,41) (30,26)

$$m = \frac{26-41}{30-10} = \frac{-15}{20} = \frac{-3}{4}$$

$4m \equiv -3 \pmod{59}$
$(15)(4m) \equiv (15)(-3) \pmod{59}$
$m \equiv -45 \pmod{59}$
$m = 14$

Plugging in (10,41)
$41 \equiv (14)(10) + k \pmod{59}$
$41 \equiv 140 + k \pmod{59}$
$-99 \equiv k \pmod{59}$
$k = 19$

So plugging in m and k, we get: $y \equiv 14x + 19 \pmod{59}$

5. Prove that in an affine plane of order n, for each line l , there are at least n -1 lines parallel to l . (Using the theorem below, and the axioms for a finite affine plane stated in problem 1)

Theorem: In an affine plane of order n, each line contains exactly n points.

Note: The theorem is to be used in the proof below. The proof is establishing that there are at least n – 1 lines parallel to l as stated above.

Proof: Let l be an arbitrary line. By [Axiom A 1], there is a point P_1 not on l. By [the provided theorem], line l has at least one point, say P_2, on it. From [Axiom A3], there is a line l' passing through P_1 and P_2. Since [P_1 is on ℓ' but not on ℓ], we have that $l' \neq l$. [Axiom A3] implies that P_2 is the only point on both l' and l. By [The provided theorem], l' has exactly [n] points on it, two of which are P_1 and P_2. Let $P_3, \ldots, P_n$ denote the remaining points on l'. For each $j \neq 2$, P_j is not on l, so that [Axiom A4] implies that there is a line l_j passing through P_j and parallel to l. Each such l_j is different from l' since [ℓ' intersects ℓ and ℓ_j is parallel to ℓ].

It follows that the lines l_j (with $j \neq 2$) are distinct by [Axiom A3] (since l' is the unique line passing through any two of the P_j's). Thus there are at least n – 1 distinct lines parallel to l (namely, the lines l_j with $j \neq 2$). □

Part II

Axiom 1. There exist 3 distinct non-collinear points.

Axiom 2. There exist 3 distinct collinear points.

Axiom 3. Given any 2 distinct points, there is exactly 1 line passing through them.

1. Justify that the axiomatic system is consistent (see appendix)

2. Justify that the axiomatic system is not complete. (see appendix)

3. Justify that the axiomatic system is independent. (see appendix)

4. What is the dual of Axiom 2?

   Dual of Axiom 2: There exist 3 distinct lines that intersect at a point.

5. Prove that there are at least 4 distinct lines, using above given axiomatic system.

Proof: By [Axiom 2], there is a line ℓ with at least 3 points on it. Call three such points P_1, P_2, and P_3. By [Axiom 1], there is at least one point Q not on ℓ. By [Axiom 3], there is exactly one line ℓ_j passing through Q and P_j for each $j \in \{1, 2, 3\}$. Each ℓ_j is not equal to ℓ because

[Q is on ℓ_j but not on ℓ].

Assume $\ell_i = \ell_j$ for some $i \neq j$ with i and j in { 1,2,3}. Then [P_i] and [P_j]

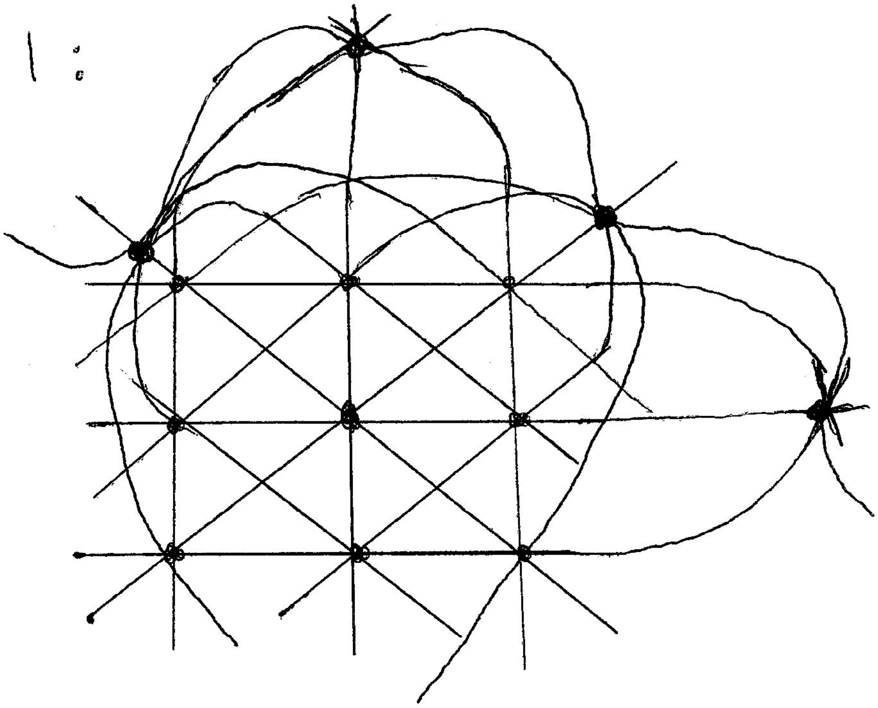
are two distinct points on two distinct lines $\boxed{\ell\ i}$ and $\boxed{\ell}$. This contradicts

$\boxed{\text{Axiom 3}}$. Hence l_1, l_2, and l_3 are all different. Since we have shown l, l_1, l_2, and l_3 are all different, the proof is complete. □

Appendix Test 1, Spring 2013

Part 1:

2)

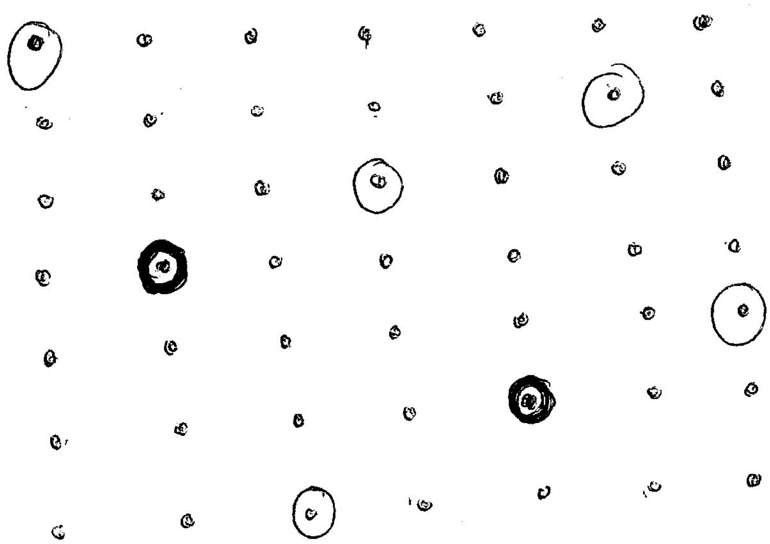


13 points

13 lines

(n^2+n+1 points and lines)

3)



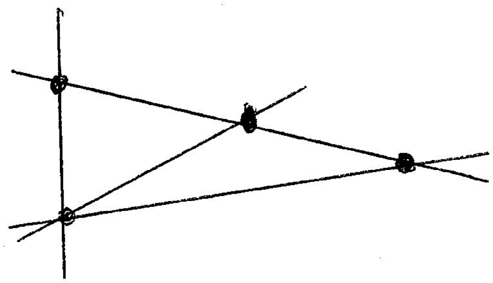
$(1,3)\ (4,1)$

$m = \frac{1-3}{4-1} = \frac{-2}{3}$

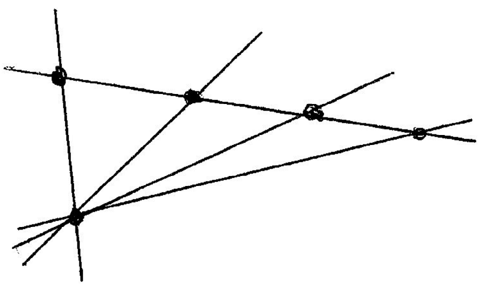
$3m \equiv -2 \pmod 7$

$m = 4$

Part II

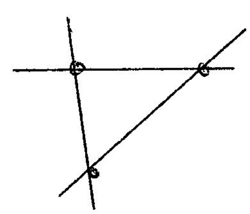
1) 

It is consistent because all three axioms in this axiomatic system hold with this model.

2) 

This axiomatic system is not complete because this model is not isomorphic with the model in number 1. It is not isomorphic because this model in number 1 has 4 points and this model in number 2 has 5 points. Both still hold for the three axioms, however.

3) Independent for Axiom 1: 

Independent for Axiom 2: 

Independent for Axiom 3: