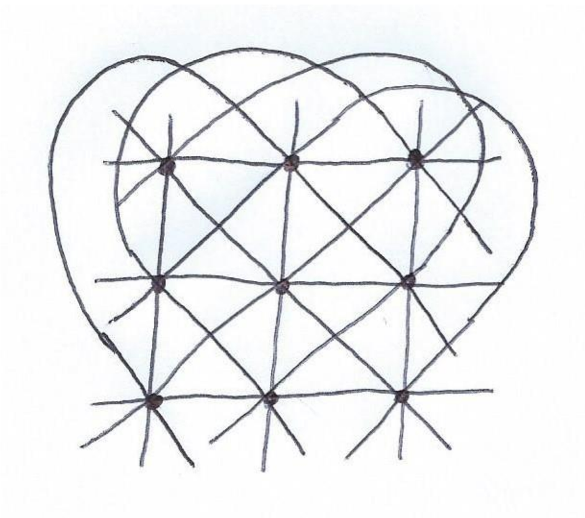
Math 532/736I: Modern Geometry

Spring 2016

Test 1: Solution Key

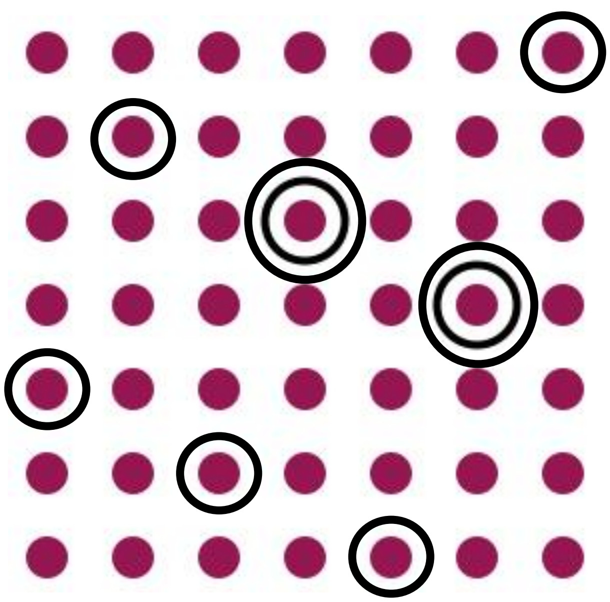
Part I

1) Affine Plane of Order 3



n^2 ***points*** & $n^2 + n$ ***lines***

2)



Starting pts: **(3, 4)** & **(5, 3)**

$m = \frac{3-4}{5-3} = -\frac{1}{2}$

$2m \equiv -1 \pmod 7$

$m \equiv 3 \pmod 7$

$y \equiv 3x + b \pmod 7$

(Plug in any point to solve)

$(4) \equiv 3(3) + b \pmod 7$

$4 \equiv 9 + 2 \pmod 7$

$4 \equiv 11 \pmod 7$

$y \equiv 3x + 2 \pmod 7$

3) 37x37 Array Starting points: (2,12) & (21,25)

$m = \frac{25-12}{21-2} = \frac{13}{19}$

$19m \equiv 13 \pmod{37}$

$m \equiv -11 \pmod{37}$

$y \equiv -11x + b \pmod{37}$

(Plug in either point to solve for b)

$(12) \equiv -11(2) + b \pmod{37}$

$12 \equiv (-22) + b \pmod{37}$

$34 \equiv b \pmod{37}$

$\boxed{y \equiv -11x + 34 \pmod{37}}$

4)

Axiom P1: There exist at least 4 points no 3 of which are collinear.
Axiom P2: There exists at least 1 line with exactly n + 1 (distinct) points on it.
Axiom P3: Given 2 distinct points, there is exactly 1 line that they both lie on.
Axiom P4: Given 2 distinct lines, there is at least 1 point on both of them.

Result: If ℓ is a line with exactly n + 1 points on it in a finite projective plane of order n and A is a point not on ℓ, then there exist at least n + 1 distinct lines passing through A.

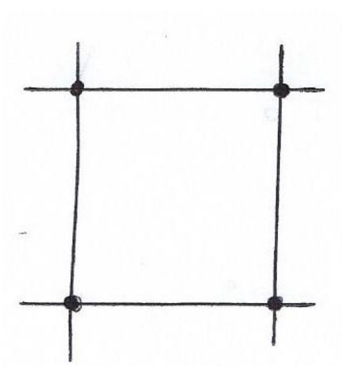
Proof. Let $P_1, P_2, \ldots, P_{n+1}$ be the $n+1$ distinct points on ℓ. Since $\boxed{\textbf{A is not on } \ell \textbf{ and each } P_j \textbf{ is on } \ell}$ we have that $A \neq P_j$ for each $j \epsilon \{1,2,\ldots,n+1\}$. By $\boxed{\textbf{Axiom P3}}$ there is a line ℓ_j passing through A and P_j for each $j \epsilon \{1,2,\ldots,n+1\}$. Since $\boxed{\textbf{A is on } \ell_j}$ and A is not on ℓ, we see that $\ell_j \neq \ell$ for each $j \epsilon \{1,2,\ldots,n+1\}$. We justify next that the $n+1$ lines $\ell_1, \ell_2, \ldots, \ell_{n+1}$ are different. Assume $\boxed{\ell_i = \ell_j}$ for some i and j in $\{1, 2, \dots, n+1\}$ with $i \neq j$. Then the two points **P_i and P_j** are both on ℓ_i. Since those two points are distinct, **Axiom P3** implies that there is **exactly one line** passing through them. Since **P_i and P_j** are two points that are both on ℓ_i and are both on ℓ, we deduce $\boldsymbol{\ell_i = \ell}$. This contradicts that $\boldsymbol{\ell_i \neq \ell}$ **for any** $\boldsymbol{i \in \{1, 2, \dots, n+1\}}$. Thus our assumption is wrong and the lines ℓ_1, $\ell_2, \dots, \ell_{n+1}$ are different. This finishes our proof that there are at least $n+1$ distinct lines passing through A. //

Part II

Axiom 1. **There exist at least one point and at least one line with the point on the line.**
Axiom 2. **Given a point, there are exactly 2 distinct lines that do not pass through the point.**
Axiom 3. **Given a line, there are exactly 2 distinct points that do not lie on the line.**

1) Justify that the axiomatic system is consistent.

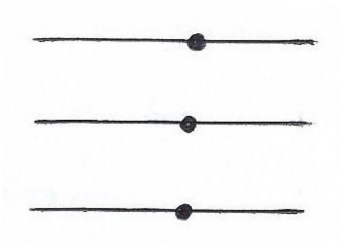
Consider the following model,



All axioms hold, thus our system is consistent.

2) Justify that the axiomatic system is *not* complete. Include some brief explanation, in complete English sentences, for your answer.

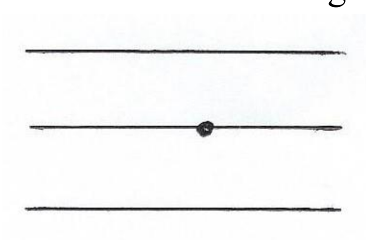
Now consider a new model,



All axioms still hold, but the previous model had 4 points and 4 lines while this model has 3 points and 3 lines. Thus, the 2 models are isomorphically different and our system is *not* complete.

3) Justify that Axiom 3 is independent of Axiom 1 and Axiom 2.

Consider the following model,



Clearly, Axiom 1 and Axiom 2 are satisfied while Axiom 3 is not. Thus, Axiom 3 is independent of Axiom 1 and Axiom 2.

4) Write the dual of each axiom below. Use correct English.

Dual of Axiom 1. There exist at least one line and at least one point with the line passing through the point.

Dual of Axiom 2. Given a line, there are exactly 2 distinct points that do not lie on the line.

Dual of Axiom 3. Given a point, there are exactly 2 distinct lines that do not pass through the point.

5) Does the principle of duality hold for this axiomatic system? Explain your answer.

Each dual is equivalent to one of the original Axioms. Thus, the principle of duality holds for this axiomatic system.

6) Fill in the boxes below to finish the proof below that, in the axiomatic system above, if there are exactly 3 points, then there are at least 3 distinct lines parallel to each other. For your convenience, the axioms are repeated here.

Axiom 1. **There exist at least one point and at least one line with the point on the line.**
Axiom 2. **Given a point, there are exactly 2 distinct lines that do not pass through the point.**
Axiom 3. **Given a line, there are exactly 2 distinct points that do not lie on the line.**

Proof. Suppose that there are exactly 3 points. We need to show that there are 3 distinct lines parallel to each other. By **Axiom 1**, there is a point P and a line ℓ_1 with P on ℓ_1. By **Axiom 3**, there are at least 2 distinct points, say Q and R, not on ℓ_1. Since P is on ℓ_1 and Q and R are not on ℓ_1, the points P, Q, and R are all distinct. Since there are only 3 points, they are P, Q, and R. By **Axiom 3**, each line passes through exactly one of P, Q, and R. By **Axiom 2**, there are exactly 2 distinct lines, say ℓ_2 and ℓ_3, that do not pass through P. Since P is on the first line, ℓ_1, ℓ_2 and ℓ_3 are distinct. Since each line passes through exactly one of P, Q, and R and P is not on ℓ_2 nor on ℓ_3, each of ℓ_2 and ℓ_3 passes through exactly one of Q and R.

Explain in the box below why ℓ_2 and ℓ_3 do not both pass through the same point.

Assume both ℓ_2 and ℓ_3 pass through Q (a similar argument works if this is R). Then no 3 of our lines (including ℓ_1) passes through R. Since the lines ℓ_1, ℓ_2, and ℓ_3 are distinct and do not pass through R, we get a contradiction to Axiom 2. This proves each of our 3 lines passes through a different point.

Thus, each of ℓ_1, ℓ_2, and ℓ_3 passes through a different point. Since each line passes through exactly one point, we deduce that ℓ_1, ℓ_2, and ℓ_3 are distinct lines which are parallel to each other. //