Math 532: Homework 7

(1) For each of the problems below, the function $f(x, y)$ is defined as follows. First f rotates (x, y) about the point $A = (1, 1)$ by $\pi/2$, then it takes the result and translates it by $B = (1, 3)$, and then it takes that result and rotates it about the point $C = (2, 4)$ by π. Thus, we can view f as being $R_{\pi,C}T_B R_{\pi/2,A}$.

(a) Calculate $f(2, 3)$.

(b) Calculate $f(3, 3)$.

(c) Calculate $f(4, 5)$.

(d) Find a point (x, y) that is mapped to itself by f. In other words, find a point (x, y) such that $f(x, y) = (x, y)$.

(2) Show that a translation by (a, b) is equivalent to a rotation about the origin by π followed by a rotation about the point $(a/2, b/2)$ by π.

(3) Using the information in the above problems, explain why the f in problem (1) is a rotation about some point D by $3\pi/2$. What are the coordinates of the point D?

Solutions

(1) For the problems, it helps to observe that

$$R_{\pi,C}T_B R_{\pi/2,A} = \begin{pmatrix} -1 & 0 & 4 \\ 0 & -1 & 8 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 5 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus,

$$R_{\pi,C}T_B R_{\pi/2,A} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} y+1 \\ -x+5 \\ 1 \end{pmatrix}.$$

We easily get then that the answer to (a) is $(4,3)$, the answer to (b) is $(4,2)$, and the answer to (c) is $(6,1)$. For (d), we want $x = y+1$ and $y = -x+5$. Thus, the answer to (d) is $(x,y) = (3,2)$.

(2) This can be done geometrically using the information in the first example on translations and rotations. We use instead the interpretation by matrices. The result follows from

$$R_{\pi,(a/2,b/2)}R_{\pi,(0,0)} = \begin{pmatrix} -1 & 0 & a \\ 0 & -1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = T_{(a,b)}.$$

(3) From (2), we can write f as a composition of 4 rotations about points by $\pi/2$, π, π, and π. Applying the theorem about compositions of rotations, we obtain that f is a rotation about some point D by $3\pi/2$. In (1) (d), we saw that $f(3,2) = (3,2)$. The only way this can occur is if $D = (3,2)$.