Math 532, 736I: Modern Geometry

Name ____________________________________

Final Exam (1993)

Show All Work
Points: Part I (100 pts), Part II (100 pts)

Part I. Each of the following is related to the first half of the course. Problem 1 is worth 14 points, Problem 2 is worth 20 points, Problem 3 is worth 12 points, Problem 4 is worth 18 points, Problems 5(a) and 5(c) are worth 8 points each, and Problems 5(b) and 5(d) are worth 10 points each.

(1) State the axioms for a finite projective plane of order n. (Number or name the axioms so you can refer to them.)

(2) You were told to know the proof of the following theorem for this exam. In the proof you give below, I will assume you are using the axioms as you have stated them in your answer to (1) (a). Your proof should depend only on these axioms and NOT on any theorems from class (so, in particular, do NOT use the principal of duality). Whenever you make use of an axiom, be sure to indicate the axiom you are using.

Theorem: *In a finite projective plane, there exist at least 4 distinct lines, no 3 of which are concurrent.*

(3) Give a model for a finite affine plane of order 3 by drawing a 3×3 array of points and drawing appropriate lines passing through these points.

(4) One way to describe the model discussed in class for an affine plane of order n is to begin with the points (a, b) in the plane where a and b are from the set $\{0, 1, \ldots, n-1\}$ and then to consider the lines $y \equiv mx + k \pmod{n}$ where m and k are from the set $\{0, 1, \ldots, n-1\}$ together with the lines $x \equiv c \pmod{n}$ where c is from the set $\{0, 1, \ldots, n-1\}$. We showed that if n is a prime, these n^2 points and $n^2 + n$ lines do in fact form a model for an affine plane of order n. For $n = 4$ (which is not a prime) these points and lines do NOT form a model for an affine plane of order n. Consider these points and lines with $n = 4$, and determine an axiom for a finite affine plane which does not hold. Justify your answer (explain why the axiom does not hold).

The axiom which I am claiming does NOT hold is: $\boxed{}$

(Refer to the information page for the axioms. In particular, your answer in the box should refer to an axiom from that page. Don't forget to justify your answer.)

(5) The next 3 pages of problems all deal with an axiomatic system consisting of the axioms below.

Axiom 1. There exist at least 3 noncollinear points.

Axiom 2. Given any 2 distinct points, there exists exactly one line passing through the 2 points.

Axiom 3. There exists at least one line with exactly 2 distinct points on it.

Axiom 4. There exists at least one line with exactly 3 distinct points on it.

(a) Explain why the axiomatic system is consistent.

(b) Explain why the axiomatic system is NOT independent.

(c) Is the axiomatic system complete? Justify your answer.

(d) Suppose we add to the axiomatic system a fifth axiom:

Axiom 5. There exist at least 5 distinct points.

Prove that there exist 2 lines which do not intersect.

Part II. Each of the following is related to the second half of the course. Problem 1 and Problem 2 are worth 16 points each, Problem 3 is worth 22 points, Problem 4 is worth 26 points, and Problem 5 is worth 20 points.

(1) Three theorems are listed on the last page of this test. They may or may not have the numbering that you are accustomed to them having from class. Prove Theorem 2 using Theorem 1.

(2) Given ΔABC, let D be the midpoint of $\overline{BC}$, let E be the midpoint of $\overline{AC}$, let F be the midpoint of $\overline{AB}$, and let G be the midpoint of $\overline{DE}$. Using vectors, prove that C, F, and G are collinear and that G is the midpoint of $\overline{CF}$.

(3) The first drawing on the second to the last page of this exam refers to this problem. Let A, B, and C be the vertices of a triangle. Let X be the center of a square drawn externally off side $\overline{AB}$, let Y be the center of a square drawn externally off side $\overline{AC}$, and let M be the midpoint of $\overline{BC}$. Let $f = R_{\pi,M} R_{\pi/2,Y} R_{\pi/2,X}$.

(a) Explain why f is the identity translation. Refer to Theorem 3 from the last page of this exam if and when you use it.

(b) Using (a), explain why ΔXMY is an isosceles right triangle.

(4) The second drawing on the second to the last page of this exam refers to this problem. The triangles ΔABC, ΔBDE, ΔDFG, ΔFHI, and ΔAHJ as shown there are equilateral. Their centers are P, Q, R, S, and T, respectively. Also, the distance from A to B is the same as the distance from F to H. Let

$$f = R_{(2\pi/3),S}R_{(2\pi/3),R}R_{(2\pi/3),Q}R_{(2\pi/3),P}.$$

Explain why $f(B) = I$. (Hint: First describe f as precisely as you can. The more precise you are – assuming you are also correct – the closer you will be to a solution and the more partial credit I can give you.)

(5) Let A, B, and C be 3 noncollinear points, and let A', B', and C' be 3 noncollinear points with $A \neq A'$, $B \neq B'$, and $C \neq C'$. Suppose that the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are parallel. Suppose further that $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ intersect at some point P, that $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ intersect at some point Q, and that $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ intersect at some point R. The next two pages contain a proof that P, Q, and R are collinear except there are some boxes which need to be filled. Complete the proof by filling in the boxes. There may be more than one correct way to fill in a box. The goal is to end up with a correct proof that P, Q, and R are collinear.

Proof: Since $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are parallel, there are real numbers k_1 and k_2 such that

$$A' - A = k_1(B' - B) = k_2(C' - C).$$

We get that

$$A - k_1 B = A' - k_1 B'.$$

We first explain why $k_1 \neq \boxed{}$.

Give explanation here: $\boxed{}$

Hence,

$$\left(\frac{1}{1-k_1}\right) A + \left(\frac{-k_1}{1-k_1}\right) B = \left(\frac{1}{1-k_1}\right) A' + \left(\frac{-k_1}{1-k_1}\right) B'.$$

By Theorem 1 (from the last page of this exam) with $t = \boxed{}$, we see that the expression on the left above is a point on line $\overleftrightarrow{AB}$ and that the expression on the right above is a point on line $\overleftrightarrow{A'B'}$. Therefore, we get that

(1) $$(1 - k_1)P = A - k_1 B.$$

Similarly, from $k_1 B - k_2 C = k_1 B' - k_2 C'$, we deduce that

(2) $$\boxed{}.$$

(Your answer should contain Q, B, and C.)

Also, from $k_2C - A = k_2C' - A'$, we deduce that

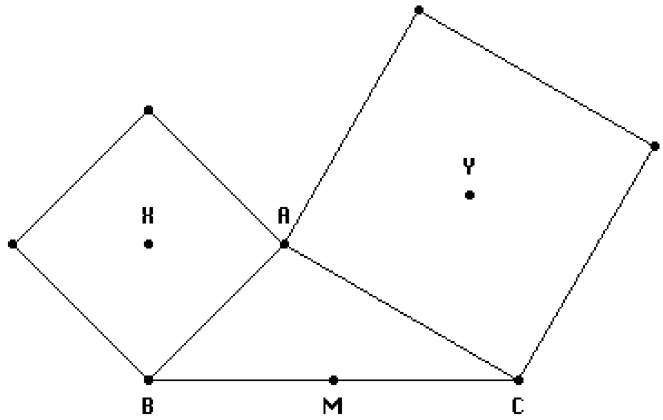
(3) $\boxed{}$.

(Your answer should contain R, A, and C.)

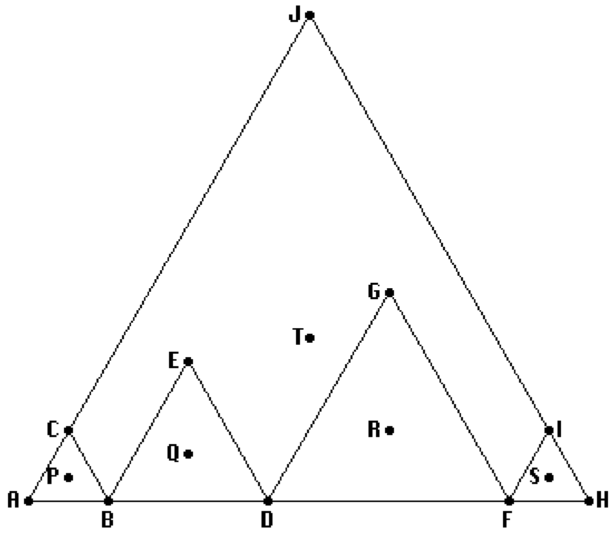
Therefore, from (1), (2), and (3),

$$(1-k_1)P + (k_1 - k_2)Q + (k_2 - 1)R = \overrightarrow{0}.$$

The result follows from Theorem $\boxed{}$ (on the last page of this test).



Part II, Problem 3



Part II, Problem 4

INFORMATION PAGE

Axiom A1. There exist at least 4 distinct points no 3 of which are collinear.

Axiom A2. There exists at least 1 line with exactly n points on it.

Axiom A3. Given any 2 distinct points, there exists exactly one line passing through the 2 points.

Axiom A4. Given any line ℓ and any point P not on ℓ, there is exactly 1 line through P that does not intersect ℓ.

Theorem 1: *Let A and B be distinct points. Then C is a point on line $\overleftrightarrow{AB}$ if and only if there is a real number t such that*

$$C = (1-t)A + tB.$$

Theorem 2: *If A, B, and C are points and there are real numbers x, y, and z not all 0 such that*

$$x + y + z = 0 \quad \text{and} \quad xA + yB + zC = 0,$$

then A, B, and C are collinear.

Theorem 3: *Let α and β be real numbers (not necessarily distinct), and let A and B be points (not necessarily distinct). If $\alpha+\beta$ is not an integer multiple of 2π, then there is point C such that $R_{\beta,B}R_{\alpha,A} = R_{\alpha+\beta,C}$. If $\alpha+\beta$ is an integer multiple of 2π, then $R_{\beta,B}R_{\alpha,A}$ is a translation.*

$$T_{(a,b)} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_{\theta,(x_1,y_1)} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & x_1(1-\cos(\theta)) + y_1\sin(\theta) \\ \sin(\theta) & \cos(\theta) & -x_1\sin(\theta) + y_1(1-\cos(\theta)) \\ 0 & 0 & 1 \end{pmatrix}$$