MATH 532, 736I: MODERN GEOMETRY

Name ___________________________________

Final Exam (1992)

Show All Work
Points: Part I (28 pts), Part II (18 pts), Part III (30 pts), Part IV (24 pts)

Part I. Each of the following is something or part of something you were asked to memorize. The first two problems in this section are worth 10 points, and the last problem is worth 8 points.

(1) Prove that if ℓ and ℓ' are two distinct lines in a finite affine plane of order n which intersect at some point P, then there does not exist a line which is parallel to both ℓ and ℓ'. For the proof, you should only use the axioms for a finite affine plane of order n and Theorem 1 (which was a homework problem) all given on the last page of this test. (Be sure to use the numbering of the axioms as given there.)

(2) Three theorems are listed on the last page of this test. They may or may not have the numbering that you are accustomed to them having from class or previous tests. Prove Theorem 3 using Theorem 2 (and not any other theorems from class).

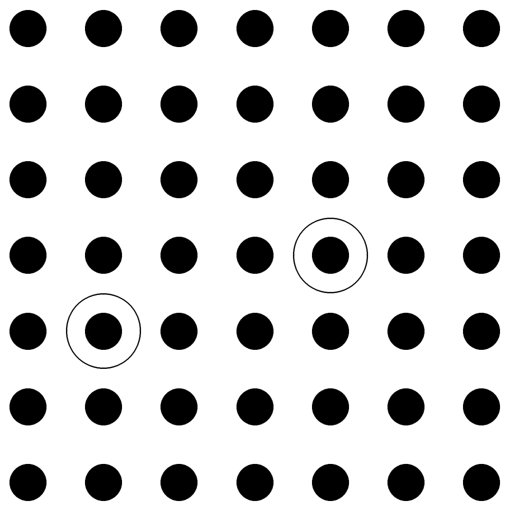
(3) On the second to the last page is a drawing consisting of 2 triangles ΔABC and $\Delta A'B'C'$ which are perspective from the line passing through X, Y, and Z. Suppose we want to prove that these triangles are also perspective from a point. Let P be the point of intersection of the lines $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$. To show that $\overleftrightarrow{CC'}$ also passes through P, we can consider two triangles which we know to be perspective from a point. The vertices of these two triangles (not necessarily the edges) and the point they are perspective from are all in the drawing. Determine the two triangles and the point they are perspective from and indicate your answers in the boxes below. (No work necessary.)

Triangles $\boxed{}$ and $\boxed{}$ are perspective from point $\boxed{}$.

(Note: You can complete the boxes to make a sentence which is true, but that will not necessarily insure that you have a correct answer. A correct answer must correspond to establishing that $\overleftrightarrow{CC'}$ passes through P as discussed above.)

Part II. Each of the following is closely related to a homework problem. Show all of the necessary steps for each problem here. The first problem is worth 6 points and the second problem is worth 12 points.

(1) Two points have been circled in the 7×7 array of points below. Using the model for a finite affine plane of order 7 discussed in class, finish circling the points that belong to the same line as the given circled points.



(2) (a) For arbitrary points A and B and arbitrary angles α, prove that $T_B R_{\alpha,A} = R_{\alpha,A+B} T_B$.

(b) Using (a), show that

$$R_{\pi/5,(1,2)} T_{(1,1)} R_{2\pi/5,(0,1)} T_{(-1,3)} R_{4\pi/5,(1,-2)} T_{(2,-1)} R_{3\pi/5,(-1,-1)}$$

is a translation T_C for some C and determine the value of C.

Part III. The problems in this section all deal with an axiomatic system consisting of the axioms below. Be sure to answer the questions being asked. For example, if you are giving a model to justify your answer in problem (1) below, make sure you also state whether your answer is, “Yes” or “No.” The first 3 problems in this section are worth 6 points each, and the last problem is worth 12 points.

Axiom 1. There exist at least 3 noncollinear points.

Axiom 2. Each line has at least 3 points on it.

Axiom 3. Given any 2 distinct points, there exists exactly one line passing through the 2 points.

Axiom 4. Given any 2 distinct lines, there is at least one point on both lines.

Axiom 5. There exists at least one point P and at least one line ℓ such that P is not on ℓ.

(1) Is the axiomatic system consistent? Justify your answer.

(2) Is the axiomatic system independent? Justify your answer.

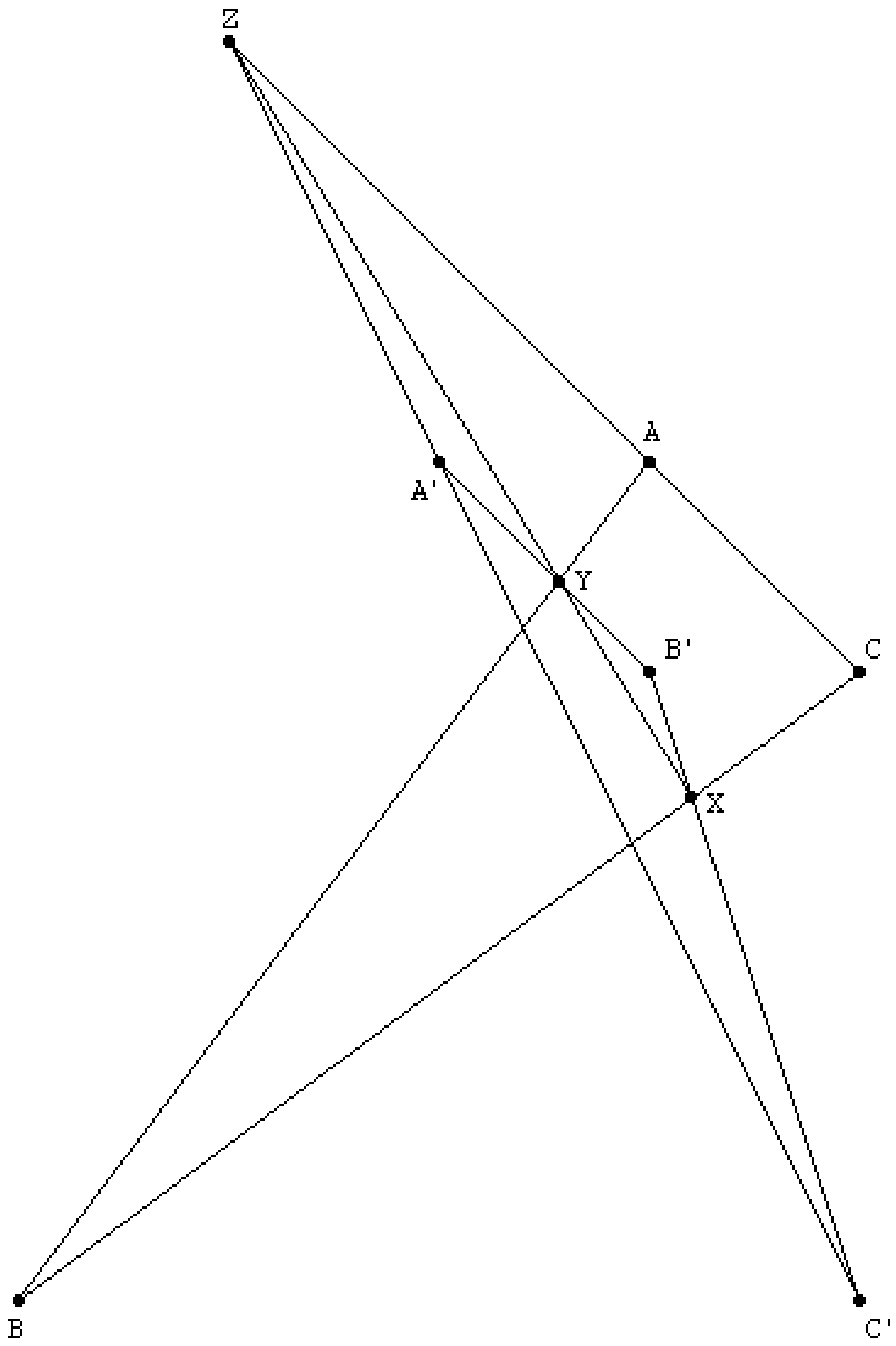
(3) Is the axiomatic system complete? Justify your answer.

(4) Does the principle of duality hold for the axiomatic system? Justify your answer.

Part IV. The first problem in this section is intended to be an application of vectors, and the second problem is intended to be an application of translations and rotations. Each is worth 12 points.

(1) Given ΔABC, let D be the midpoint of $\overline{BC}$, let E be the midpoint of $\overline{AC}$, let F be the midpoint of $\overline{AB}$, and let G be the midpoint of $\overline{DE}$. Prove that C, F, and G are collinear and that G is the midpoint of $\overline{CF}$.

(2) Let $P_1, P_2, \ldots, P_n$ be n (not necessarily distinct) points in the plane. Let A and B be two arbitrary points. Beginning with $A_0 = A$, for $j \in \{1, 2, \ldots, n\}$, define A_j as the point you get by rotating A_{j-1} about P_j by π. Beginning with $B_0 = B$, for $j \in \{1, 2, \ldots, n\}$, define B_j as the point you get by rotating B_{j-1} about P_j by π. Prove that A, B, A_n, and B_n either all lie on a line or they are the vertices of a parallelogram. (Comment: You may want to consider even and odd n separately.)



Part I, Problem 3

INFORMATION PAGE

Axiom A1. There exist at least 4 distinct points no 3 of which are collinear.

Axiom A2. There exists at least 1 line with exactly n points on it.

Axiom A3. Given any 2 distinct points, there exists exactly one line passing through the 2 points.

Axiom A4. Given any line ℓ and any point P not on ℓ, there is exactly 1 line through P that does not intersect ℓ.

Theorem 1: *In a finite affine plane of order n, each line contains exactly n points.*

Theorem 2: *Let A and B be distinct points. Then C is a point on line $\overleftrightarrow{AB}$ if and only if there is a real number t such that*

$$C = (1-t)A + tB.$$

Theorem 3: *If A, B, and C are points and there are real numbers x, y, and z not all 0 such that*

$$x + y + z = 0 \quad \text{and} \quad xA + yB + zC = 0,$$

then A, B, and C are collinear.

$$T_{(a,b)} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_{\theta,(x_1,y_1)} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & x_1(1-\cos(\theta)) + y_1\sin(\theta) \\ \sin(\theta) & \cos(\theta) & -x_1\sin(\theta) + y_1(1-\cos(\theta)) \\ 0 & 0 & 1 \end{pmatrix}$$