Math 241: Quiz 4

Show ALL Work Name **Solutions**

1. Find parametric equations for the line through the point $(5, 4, -7)$ that is parallel to the line given by $x = t$, $y = -3 - t$ and $z = 4 + 2t$. SHOW WORK (even if you can do the work without showing it).

   Parametric Equations: $\boxed{\begin{aligned} x &= 5 + t \\ y &= 4 - t \\ z &= -7 + 2t \end{aligned}}$

   A vector going in the direction of the given line is $\overrightarrow{v} = \langle 1, -1, 2\rangle$. So this vector is in the direction of the parallel line as well. Since $(5, 4, -7)$ is a point on the parallel line, the above answer follows.

2. Find parametric equations for the line through the point $(5, 4, -7)$ that is perpendicular to the line $x = t$, $y = -3 - t$ and $z = 4 + 2t$ and intersects this line. Be sure to show your work.

   Parametric Equations: $\boxed{\begin{aligned} x &= 5 + 3t \\ y &= 4 + t \\ z &= -7 - t \end{aligned}}$

   Solution 1. The vector $\overrightarrow{n} = \langle 1, -1, 2\rangle$ is in the direction of the given line and is therefore a normal for a plane perpendicular to the given line. This means that the plane $\mathcal{P}$ perpendicular to the given line and passing through $(5, 4, -7)$ has the equation $x - y + 2z = -13$ (where the -13 came from plugging in the point). The value of t where the given line intersects the plane $\mathcal{P}$ satisfies

   $$t - (-3 - t) + 2(4 + 2t) = -13 \quad \text{or, equivalently,} \quad 6t = -24.$$

   So $t = -4$ where the given line intersects $\mathcal{P}$. Thus, the point $(-4, 1, -4)$ (gotten by plugging in $t = -4$ into the parametric equations for the given line) is on the given line and on the plane $\mathcal{P}$ perpendicular to the given line and passing through $(5, 4, -7)$. Therefore, the line we want passes through both $(-4, 1, -4)$ and $(5, 4, -7)$. A vector going from the first of these to the second is $\overrightarrow{v} = \langle 9, 3, -3\rangle = 3\langle 3, 1, -1\rangle$. The answer follows from using that $(5, 4, -7)$ is a point on the perpendicular line and $\langle 3, 1, -1\rangle$ is the direction of this line. ■

(see second page for two more solutions)

Solution 2. The given line passes through the point $P = (0, -3, 4)$ and is parallel to the vector $\overrightarrow{v} = \langle 1, -1, 2 \rangle$. Let $Q = (5, 4, -7)$. Then $\overrightarrow{PQ} = \langle 5, 7, -11 \rangle$ and $\overrightarrow{v}$ are perpendicular to

$$\overrightarrow{PQ} \times \overrightarrow{v} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 5 & 7 & -11 \\ 1 & -1 & 2 \end{vmatrix} = \langle 3, -21, -12 \rangle.$$

This vector and $\overrightarrow{v}$ are both perpendicular to a vector going in the direction of the line we want passing through Q. Therefore, to get a vector going in the direction of the line we want, we compute

$$\overrightarrow{PQ} \times \overrightarrow{v} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 3 & -21 & -12 \\ 1 & -1 & 2 \end{vmatrix} = \langle -54, -18, 18 \rangle = -18 \langle 3, 1, -1 \rangle.$$

Since $(5, 4, -7)$ is a point on the perpendicular line we want and $\langle 3, 1, -1 \rangle$ is in the direction of this line, the answer follows. ■

Solution 3. The given line ℓ passes through the point $P = (0, -3, 4)$ and is parallel to the vector $\overrightarrow{v} = \langle 1, -1, 2 \rangle$. Let $Q = (5, 4, -7)$. We project the vector $\overrightarrow{PQ} = \langle 5, 7, -11 \rangle$ on to $\overrightarrow{v}$ to get

$$\text{proj}_{\overrightarrow{v}} \overrightarrow{PQ} = \frac{\overrightarrow{v} \bullet \overrightarrow{PQ}}{\| \overrightarrow{v} \|^2} \overrightarrow{v} = \frac{-24}{6} \overrightarrow{v} = -4 \langle 1, -1, 2 \rangle = \langle -4, 4, -8 \rangle.$$

The point $R = (a, b, c)$ on ℓ where the line $\overleftrightarrow{QR}$ is perpendicular to ℓ is given by

$$\langle a, b, c \rangle = \langle 0, -3, 4 \rangle + \langle -4, 4, -8 \rangle = \langle -4, 1, -4 \rangle.$$

So the line we want is the line passing through $Q = (5, 4, -7)$ and $R = (-4, 1, -4)$. Since this line passes through $Q = (5, 4, -7)$ and is in the direction of $\overrightarrow{QR} = \langle -9, -3, 3 \rangle = -3 \langle 3, 1, -1 \rangle$, the answer follows. ■