Math 241: Quiz 3 SOLUTIONS

1. Let $A = (2, 1, -3)$, and let $\mathcal{P}$ be the plane given by $x + y - z = 0$. Calculate the point B on the plane $\mathcal{P}$ that is nearest to A. Simplify your answer.

   Point B: $\boxed{(0, -1, -1)}$

   Solution 1: First, we find parametric equations for a line ℓ perpendicular to the plane $\mathcal{P}$ that passes through A. Since a normal to the plane is $\langle 1, 1, -1\rangle$, this vector is parallel to (in the direction of) ℓ. Since ℓ goes through A, parametric equations for ℓ are given by $x = 2 + t$, $y = 1 + t$ and $z = -3 - t$. The point B is the point $(2 + t, 1 + t, -3 - t)$ on ℓ which is also on $\mathcal{P}$. Since $\mathcal{P}$ is given by $x + y - z = 0$, we want

   $$(2 + t) + (1 + t) - (-3 - t) = 0 \quad \text{or, equivalently,} \quad 6 + 3t = 0.$$

   This implies $t = -2$, so the point B is $(2 - 2, 1 - 2, -3 - (-2)) = (0, -1, -1)$. ■

   Solution 2: The point $Q = (0, 0, 0)$ is on the plane $\mathcal{P}$ (any point Q on $\mathcal{P}$ can be used here). We compute the projection of the vector $\overrightarrow{QA} = \langle 2, 1, -3\rangle$ onto the normal $\overrightarrow{n} = \langle 1, 1, -1\rangle$ to plane $\mathcal{P}$. This is given by

   $$\text{proj}_{\overrightarrow{n}}\,\overrightarrow{QA} = \frac{\overrightarrow{n} \cdot \overrightarrow{QA}}{\|\overrightarrow{n}\|^2}\,\overrightarrow{n} = \frac{6}{\sqrt{3}^2}\,\langle 1, 1, -1\rangle = 2\,\langle 1, 1, -1\rangle = \langle 2, 2, -2\rangle.$$

   We want then a point B such that $\overrightarrow{BA} = \langle 2, 2, -2\rangle$. Since $A = (2, 1, -3)$, we deduce $B = (2 - 2, 1 - 2, -3 - (-2)) = (0, -1, -1)$. ■

2. The two planes given by $x - 2y + z = 4$ and $2x + y - 2z = 5$ intersect in a line ℓ. Find the parametric equations for the line ℓ' which is parallel to ℓ and passes through the point $(1, 1, 0)$.

   Line: $\boxed{\begin{array}{l} x = 1 + 3t \\ y = 1 + 4t \\ z = 5t \end{array}}$

   Solution: The normals to the planes, given by $\overrightarrow{n_1} = \langle 1, -2, 1\rangle$ and $\overrightarrow{n_2} = \langle 2, 1, -2\rangle$, are both perpendicular to a vector parallel to ℓ and, hence, parallel to ℓ'. So a vector perpendicular to both $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ will be parallel to ℓ'. We can find such a vector by computing

   $$\overrightarrow{n_1} \times \overrightarrow{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \langle 3, 4, 5\rangle.$$

   Since $(1, 1, 0)$ is on ℓ', parametric equations for ℓ' are given by $x = 1 + 3t$, $y = 1 + 4t$ and $z = 5t$. ■