Math 241: Quiz 11 Fall, 2019 SOLUTIONS

1. Calculate the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 1$ and below by the cone $4z = 3\sqrt{x^2 + y^2}$. Justify your answer and simplify it. In particular, your answer should not involve any trigonometric or inverse trigonometric functions.

Volume: $\boxed{\dfrac{2\pi}{15}}$ (simplify)

Solution 1: In spherical coordinates, the sphere is given by $\rho = 1$ and the cone is $\phi = \phi_0$ for some angle ϕ_0. To compute ϕ_0, we rewrite the equation of the cone $4z = 3\sqrt{x^2 + y^2}$ as

$$4\rho\cos\phi_0 = 4z = 3r = 3\rho\sin\phi_0.$$

This can be rewritten as $\tan\phi_0 = 4/3$. One draws a picture of a right triangle with $\tan\phi_0 = 4/3$, where ϕ_0 is one of the acute angles, to see that $\sin\phi_0 = 4/5$ (not important below) and $\cos\phi_0 = 3/5$ (important below). Then we see that the volume is

$$\int_0^{2\pi}\int_0^{\phi_0}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\phi_0} \frac{\rho^3}{3}\bigg|_0^1 \sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\phi_0} \sin\phi\,d\rho\,d\phi\,d\theta = \frac{1}{3}\int_0^{2\pi}(-\cos\phi)\bigg|_0^{\phi_0} d\phi\,d\theta$$

$$= \frac{1}{3}\big(-\cos\phi_0 + 1\big)\int_0^{2\pi} d\theta = \frac{2\pi}{3}\big(-\cos\phi_0 + 1\big)$$

$$= \frac{2\pi}{3}\left(-\frac{3}{5} + 1\right) = \frac{4\pi}{15}. \ \blacksquare$$

Solution 2: In cylindrical coordinates, the sphere can be written as $z = \sqrt{1 - r^2}$ and the cone as $z = (3/4)r$. These intersect in a circle. Specifically, they intersect when $\sqrt{1 - r^2} = (3/4)r$. Squaring, this becomes $1 - r^2 = (9/16)r^2$. Solving for r, we get $r = 4/5$ (note that r is always ≥ 0). In the xy-plane, we want points (x, y) that lie on or inside the circle given by $r = 4/5$. Hence, the volume is

$$\int_0^{2\pi}\int_0^{4/5}\int_{(3/4)r}^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta.$$

This may be a little more difficult to integrate than the triple integral in spherical coordinates, but it is reasonably doable. You should get the same answer as above. $\blacksquare$