Math 241: Final Exam

Name ___________________________________

Instructions and Point Values: Put your name in the space provided above. Check that your test contains 11 different pages including one blank page. Work each problem below and show ALL of your work. Unless stated otherwise, you do not need to simplify your answers. Do NOT use a calculator.

There are 100 total points possible on this exam. The points for each problem in each part is indicated below.

PART I:

Problem (1) is worth 8 points.

Problem (2) is worth 7 points.

Problem (3) is worth 7 points.

Problem (4) is worth 7 points.

Problem (5) is worth 8 points.

Problem (6) is worth 7 points.

Problem (7) is worth 8 points.

PART II:

Problem (1) is worth 12 points.

Problem (2) is worth 12 points.

Problem (3) is worth 12 points.

Problem (4) is worth 12 points.

PART I.

(1) Given the vectors $\vec{u} = \langle -1, 2, -2 \rangle$ and $\vec{v} = \langle -1, 0, 1 \rangle$, compute each of the following:

(a) $\vec{v} - 2\vec{u}$

Answer: $\boxed{}$

(b) $\vec{u} \cdot \vec{v}$ (the dot product of $\vec{u}$ and $\vec{v}$)

Answer: $\boxed{}$

(c) $\vec{u} \times \vec{v}$ (the cross product of $\vec{u}$ and $\vec{v}$)

Answer: $\boxed{}$

(2) (a) Calculate the directional derivative of $f(x, y) = x^3 - 6y^2x$ at the point $(1, 1)$ in the direction of the vector $\langle 6, 8 \rangle$.

Answer: $\boxed{}$

(b) In what direction is the directional derivative of $f(x, y) = x^3 - 6y^2x$ *minimized* at the point $(1, 1)$? Your answer should be a vector.

Answer: $\boxed{}$

(3) There are two points P and Q on the graph of $x^2 - y^2 = z^2 - 1$ such that the tangent plane at P and the tangent plane at Q are both parallel to the xz-plane. Determine the points P and Q. (There are different good ways to do this problem. Any correct method is fine, but make sure that your work justifies your answer.)

Two Points: $\boxed{}$ and $\boxed{}$

(4) Calculate the line integral $\int_{\mathcal{C}} (x + 2y)\,\mathrm{d}x + (x - 2y)\,\mathrm{d}y$ where $\mathcal{C}$ is the line segment from $(1, 1)$ to $(3, -1)$.

Answer: $\boxed{}$

(5) Using Lagrange multipliers, find the maximum value and the minimum value of $f(x, y) = x^3 y$ given the constraint $3x^4 + y^4 = 1$.

Maximum Value: $\boxed{}$

Minimum Value: $\boxed{}$

(6) Calculate $\iiint\limits_S (x^2+y^2)^{1/2}\,\mathrm{d}V$ where S is the solid between the paraboloids $z = x^2+y^2-1$ and $z = -2x^2 - 2y^2 + 8$.

Answer: $\boxed{}$

(7) Let $f(x, y) = x^2y - xy^2 + 3xy$. Determine whether each point below is the location of a local maximum, a local minimum, a saddle point, or not a critical point (so you are to choose one of four possibilities for each point). Be sure to justify your answers.

(a) $(0, 0)$

Answer: $\boxed{}$

(b) $(1, -1)$

Answer: $\boxed{}$

(c) $(-1, 1)$

Answer: $\boxed{}$

PART II.

(1) Using Green's Theorem, calculate

$$\int_{\mathcal{C}} (yx + 3x^2 \sin y + 3y)\,\mathrm{d}x + (x^3 \cos y - y^3 + 3x)\,\mathrm{d}y,$$

where $\mathcal{C}$ is the square oriented counter-clockwise with vertices at $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$.

Answer: $\boxed{}$

(2) Let ℓ denote the line given by the parametric equations $x = 1 - t$, $y = -1 - t$, and $z = 2$ (so any value of t produces a point (x, y, z) on line ℓ). Let ℓ' denote the line given by the parametric equations $x = -1 + s$, $y = 3 + s$, and $z = -1 + 3s$ (so any value of s produces a point (x, y, z) on line ℓ'). The purpose of this problem is to have you determine the point P on ℓ and the point Q on ℓ' with the distance from P to Q as small as possible. This distance is the distance between two lines. But this problem is not asking for that distance. Instead, we are finding the points P and Q.

(a) Write a function D of t and s that represents the square of the distance from a point on the line ℓ to a point on the line ℓ'.

$D =$ ☐

(b) Find the values of t and s that minimize the distance from a point on the line ℓ to a point on the line ℓ'.

$t =$ ☐

$s =$ ☐

(c) What are the points P and Q mentioned above before part (a)?

Two Points: ☐ and ☐

(3) Calculate each of the following:

(a) $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{0} \sqrt{x^2+y^2+16}\,\mathrm{d}x\,\mathrm{d}y$

Answer: $\boxed{}$

(b) $\displaystyle\int_{-1}^{1}\int_{0}^{x+1} (4y-y^2)^{3/2}\,\mathrm{d}y\,\mathrm{d}x$

Answer: $\boxed{}$

(4) Let $\mathcal{P}$ be the plane $x-2y-z=3$. The points $Q=(3,-2,4)$ and $R=(4,-1,3)$ are both on the plane $\mathcal{P}$. There are two planes $\mathcal{P}'$ and $\mathcal{P}''$ each intersecting plane $\mathcal{P}$ at a 60° angle and each passing through both Q and R. The purpose of this problem is to find equations for the planes $\mathcal{P}'$ and $\mathcal{P}''$.

(a) If the plane $ax+by+cz=d$ is either $\mathcal{P}'$ or $\mathcal{P}''$ so that it intersects $\mathcal{P}$ at a 60° angle (that is the smallest angle between these two planes is 60°), explain why

$$a-2b-c=\pm\frac{\sqrt{6}}{2}\sqrt{a^2+b^2+c^2}.$$

Work and Explanation:

(b) The points (x,y,z) that satisfy $ax+by+cz=d$ are the same as the points (x,y,z) that satisfy $atx+bty+ctz=dt$ for any non-zero number t (in other words, if we multiply both sides of an equation for a plane by a non-zero constant, then the new equation still describes the same plane). This implies that we can alter the coefficients of x, y, and z describing the plane $ax+by+cz=d$ so that the sum of the squares of the coefficients is 6 (or whatever else we want). Suppose then that $a^2+b^2+c^2=6$. Show why if the plane $ax+by+cz=d$ is as in part (a), then

$$a-2b-c=\pm 3.$$

(There are a lot of words here, but don't let that deceive you. This part should take very little work.)

Very Little work:

(c) The vector $\overrightarrow{QR}$ is related to the plane $ax + by + cz = d$ of part (a). Explain why

$$a + b - c = 0$$

using the vector $\overrightarrow{QR}$. (In addition to the mathematics, write an English sentence that explains the connection between $\overrightarrow{QR}$ and the plane $ax + by + cz = d$.)

Work and Explanation:

(d) Use the equations $a - 2b - c = 3$, $a + b - c = 0$, and $a^2 + b^2 + c^2 = 6$ from parts (b) and (c) to determine two possibilities for (a, b, c) (where again the plane $ax + by + cz = d$ is as in part (a)).

Two Points: $\boxed{}$ and $\boxed{}$

(e) What are equations for the planes $\mathcal{P}'$ and $\mathcal{P}''$ in the discussion before part (a)?

Equations of Planes: $\boxed{}$ and $\boxed{}$